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The upper connected edge geodetic number of a graph

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Abstract

For a non-trivial connected graph G, a set $S \subseteq V(G)$ is called an edge geodetic set of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an edge geodetic basis. A connected edge geodetic set of G is an edge geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected edge geodetic set of G is the connected edge geodetic number of G and is denoted by $g_{1c}(G)$. A connected edge geodetic set of cardinality $g_{1c}(G)$ is called a g_{1c}- *set* of G or connected edge geodetic basis of G. A connected edge geodetic set S in a connected graph G is called a minimal connected edge geodetic set if no proper subset of S is a connected edge geodetic set of G. The upper connected edge geodetic number $g^+_{1c}(G)$ is the maximum cardinality of a minimal connected edge geodetic set of G. Graphs G of order p for which $g_{1c}(G) = g^+_{1c} = p$ are characterized. For positive integers r,d and $n \geq d + 1$ with $r \leq d \leq 2r$, there exists a connected graph of radius r, diameter d and upper connected edge geodetic number n. It is shown for any positive integers $2 \leq a < b \leq c$, there exists a connected graph G such that $g_1(G) = a, g_{1c}(G) = b$ and $g^+_{1c}(G) = c$.

1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [4]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. It is known that this *distance* is a metric on the vertex set $V(G)$. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. For a vertex v of G, the *eccentricity* $e(v)$ is the *distance* between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G,

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and the maximum eccentricity is its *diameter, diam* G of G. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a *g-set* or *geodetic basis*. The geodetic number of a graph was introduced in [1, 5] and further studied in [2,3]. It was shown in [5] that determining the geodetic number of a graph is NP-hard problem. The connected geodetic number was studied by Santhakumaran et al. in [8]. A *connected geodetic set* of G is a geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the *connected geodetic number* G and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a g_c- set of G or a *connected geodetic basis* of G. The upper connected geodetic number and the forcing connected geodetic number of a graph was introduced and studied by Santhakumaran et al. in [9]. A connected geodetic set S in a connected graph G is called a *minimal connected geodetic set* if no proper subset of S is a connected geodetic set of G. The *upper connected geodetic number* $g_c^+(G)$ is the maximum cardinality of a minimal connected geodetic set of G.

The edge geodetic number of a graph was studied by Santhakumaran and John in [7]. A set $S \subseteq V(G)$ is called an *edge geodetic set* of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its *edge geodetic sets* and any edge geodetic set of order $g_1(G)$ is an *edge geodetic basis*. The upper edge geodetic number and the forcing edge geodetic number of a graph was introduced and studied by Santhakumaran and John in [10]. A *connected edge geodetic set* of G is an edge geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a *connected edge geodetic set* of G is the *connected edge geodetic number* of G and is denoted by $g_{1c}(G)$. A connected edge geodetic set of cardinality $g_{1c}(G)$ is called a g_{1c}- set of G or *connected edge geodetic basis* of G. The connected edge geodetic number of a graph was intoduced and studied by Santhakumaran and John in [11]. Although the edge geodetic number is greater than or equal to the geodetic number for an arbitrary graph, the properties of the edge geodetic sets and results regarding edge geodetic number are quite different from that of geodetic concepts. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle when considering geodetic sets, some of the edges may be left out. This drawback is rectified in the case of edge geodetic sets and hence considering edge geodetic sets is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to tranport the commodities to important places. This is the motivation behind the introduction and study of edge geodetic concepts. The following are some of the main results proved in [11]. Connected graphs of order p with connected edge geodetic number 2 are characterized. Various necessary conditions for the connected edge geodetic number of a graph to be $p-1$ or p are given. It is shown that every pair k, p of integers with $3 \leq k \leq p$ is realizable as the connected edge geodetic number and order of some connected graph. For positive integers r, d and $n \geq d+1$ with $r \leq d \leq 2r$, there exists a connected graph of radius r, diameter d and connected edge geodetic number n. If p, d and n are integers such that $2 \leq d \leq p-1$ and $d+1 \leq n \leq p$, then there exists a connected graph G of order p, diameter d and $g_{1c}(G) = n$. Also if p, a and b are positive integers such that $2 \leq a < b \leq p$, then there exists a connected graph G of order p, $g_1(G) = a$ and $g_{1c}(G) = b$.

In this paper, we introduced and study the uper connected edge geodetic nmber and the forcing connected connected edge geodetic number of a graph. Throughout this paper, G denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

Theorem 1.1. [7] *Each extreme vertex of a connected graph G belongs to every edge geodetic set of G.*

Theorem 1.2. [7] *For any non-trivial tree T with k end vertices, $g_1(T) = k$.*

Theorem 1.3. [11] *Each cut vertex of a connected graph G belongs to every connected edge geodetic set of G.*

Theorem 1.4. [11] *For a connected graph G, $g_{1c}(G) \geq 1+$ diam G.*

2 The Upper Connected Edge Geodetic Number of a Graph

Definition 2.1. A connected edge geodetic set S in a connected graph G is called a *minimal connected edge geodetic set* if no proper subset of S is a connected edge geodetic set of G. The *upper connected edge geodetic number* $g_{1c}^+(G)$ is the maximum cardinality of a *minimal connected edge geodetic set* of G.

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1, v_3, v_4\}$ is a connected edge geodetic basis of G so that $g_{1c}(G) = 3$ and $S' = \{v_2, v_3, v_4, v_5\}$ is a connected edge geodetic set of G and it is clear that no proper subset of S' is a connected edge geodetic set of G and so S' is a minimal connected edge geodetic set of G. Since $|V(G)| = 5$, it follows that $g_{1c}^+(G) = 4$.

Every minimum connected edge geodetic set of G is a minimal connected edge geodetic set of G. The converse is not true. For the graph G given in Figure 2.1, $S' = \{v_2, v_3, v_4, v_5\}$ is a minimal connected edge geodetic set and is not a minimum connected edge geodetic set of G.

Proposition 2.3. *For a connected graph G of order p, $2 \leq g_{1c}(G) \leq g_{1c}^+(G) \leq p$.*

Proof. Any connected edge geodetic set needs at least two vertices and so $g_{1c}(G) \geq 2$. Since every minimum connected edge geodetic set is a minimal connected edge geodetic set, $g_{1c}(G) \leq g_{1c}^+(G)$. Also, since $V(G)$ induces a connected edge geodetic set of G, it is clear that $g_{1c}^+(G) \leq p$. Thus $2 \leq g_{1c}(G) \leq g_{1c}^+(G) \leq p$. □

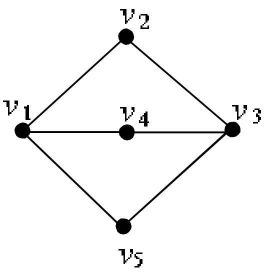

Figure 2.1: G

For the complete graph K_2, $g_{1c}(K_2) = 2$ and for any non-trivial tree T of order p, $g^+_{1c}(T) = p$. Thus the bounds in Proposition 2.3 are sharp. Also, all the inequalities in Proposition 2.3 are strict. For the graph G given in Figure 2.1, $g_{1c}(G) = 3$, $g^+_{1c}(G) = 4$, $p = 5$ so that $2 < g_{1c}(G) < g^+_{1c}(G) < p$.

Now we introduce a semi-extreme vertex of a graph and proceed to characterize graphs of order p for which $g_{1c}(G) = p$.

Definition 2.4. A vertex v in a connected graph G is said to be *semi-extreme* vertex of G if $\Delta(< N(v) >) = |N(v)| - 1$, where Δ denotes the maximum degree of a graph.

Remark 2.5. Every extreme vertex of G is a semi-extreme vertex of G. The converse is not true. For the graph G given in Figure 2.2, v_1, v_3, v_4 and v_5 are the semi-extreme vertices of G, whereas v_3 and v_4 are not extreme vertices of G.

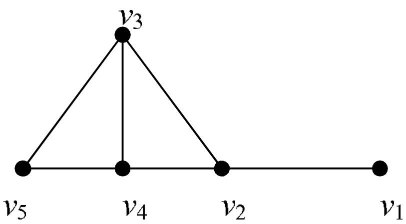

Figure 2.2: G

Theorem 2.6. Each semi-extreme vertex of a graph G belongs to every edge geodetic set of G.

Proof. Let S be an edge geodetic set of G. Suppose that there exists a semi-extreme vertex u such that $u \notin S$. Since $\Delta(< N(u) >) = |N(u)| - 1$, there exists $v \in N(u)$ such that $deg_{<N(u)>}(v) = |N(u)| - 1$. Since S is an edge geodetic set of G, the edge $e = uv$ lies on a $x - y$ geodesic $P : x = x_0, x_1, \ldots, x_i = u, x_{i+1} = v, \ldots, x_n = y$ with $x, y \in S$. Then $u \neq x, y$. Since $deg_{<N(u)>}(v) = |N(u)| - 1$, it follows that v is adjacent to x_{i-1}, which is a contradiction to the fact that P is a $x - y$ geodesic. Hence each edge geodetic set contains all the semi-extreme vertices. $\square$

Now, we observe that Theorem 1.1 is a corollary to Theorem 2.6.

Corollary 2.7. For a connected graph G with k semi-extreme vertices and l cut vertices, $g^{+}_{1c}(G) \geq \max\{2, k+l\}$.

Proof. This follows from Theorems 1.3 and 2.6. □

The following theorem characterizes graphs G of order p for which $g^{+}_{1c}(G) = p$.

Theorem 2.8. For a connected graph G of order p, the following are equivalent:

(i) $g^{+}_{1c}(G) = p$

(ii) $g_{1c}(G) = p$

(iii) Every vertex of G is either a cut vertex or a semi-extreme vertex of G.

Proof. (i) implies (ii). Let $g^{+}_{1c}(G) = p$. Then $S = V(G)$ is the unique minimal connected edge geodetic set of G. Since no proper subset of S is a connected edge geodetic set, it is clear that S is the unique minimum connected edge geodetic set of G and so $g_{1c}(G) = p$.
(ii) implies (iii). Let $g_{1c}(G) = p$. Let v be a vertex of G such that v is neither a cut vertex of G nor a semi-extreme vertex of G. Since v is not a semi-extreme vertex of G, for every $u \in N(v)$, there exists $x \in N(v) - \{u\}$ such that $ux \notin E(G)$. Let $S = V(G) - \{v\}$. The edge uv lies on the geodesic u, v, x and it follows that S is an edge geodetic set of G. Also, since v is not a cut vertex of G, $< S >$ is connected and so S is a connected edge geodetic set of G. Hence $g_{1c}(G) \leq p-1$, which is a contradiction.
(iii) implies (i). This follows from Corollary 2.7. □

Corollary 2.9.

(i) *For the complete graph* $K_p (p \geq 2), g_{1c}(K_p) = g^{+}_{1c}(K_p) = p$.

(ii) *For any tree* T *of order* p, $g_{1c}(T) = g^{+}_{1c}(T) = p$.

Proposition 2.10. *If* G *is a connected graph of order* p *with* $g_{1c}(G) = p-1$, *then* $g^{+}_{1c}(G) = p-1$.

Proof. Since $g_{1c}(G) = p-1$, it follows from Proposition 2.3 that $g^{+}_{1c}(G) = p$ or $p-1$. If $g^{+}_{1c}(G) = p$, then by Theorem 2.8, $g_{1c}(G) = p$, which is a contradiction. Hence $g^{+}_{1c}(G) = p-1$. □

Remark 2.11. The converse of the Proposition 2.10 is false. For the graph G given in Figure 2.1, $g^{+}_{1c}(G) = 4 = p-1$ and $g_{1c}(G) = 3 = p-2$.

It is proved in [11] that for the cycle C_p, $g_{1c}(C_p) = \begin{cases} \frac{p}{2}+1 & \text{if } p \text{ is even} \\ \lceil \frac{p}{2} \rceil + 2 & \text{if } p \text{ is odd.} \end{cases}$
For the cycle C_p, the same result is also true for $g^{+}_{1c}(C_p)$.

Theorem 2.12. *For the cycle C_p, $g^+_{1c}(C_p) = \begin{cases} \frac{p}{2}+1 & \textit{if } p \textit{ is even} \\ \lfloor \frac{p}{2} \rfloor + 2 & \textit{if } p \textit{ is odd.} \end{cases}$*

Proof.
Case 1. p is even. Let $p = 2n$. Let $C_{2n} : v_1, v_2, v_3, ..., v_{2n}, v_1$ be the cycle of order $2n$. Let $S = \{v_1, v_2, v_3, ..., v_{n+1}\}$. It is clear that S is a connected edge geodetic set of C_p. We prove that S is a minimal connected edge geodetic set of C_p. Let S' be a proper subset of S. Then there is a vertex, say $x \in S$ such that $x \notin S'$. If $x = v_1$ or v_{n+1}, then S' is not an edge geodetic set of C_p. If $x \neq v_1, v_{n+1}$, then the subgraph of G induced by S' is not connected. Hence it follows that S is a minimal connected edge geodetic set of C_p and so $g^+_{1c}(G) \geq |S| = n + 1$.

Now, we claim that $g^+_{1c}(G) = n + 1$. Otherwise, there is a minimal connected edge geodetic set W such that $|W| = m > n + 1$. Since W is a connected edge geodetic set of C_p, the subgraph induced by W is a path say $v_{i+1}, v_{i+2}, ..., v_{i+m}$. It is clear that $T = \{v_{i+1}, ..., v_{i+n+1}\}$ is a connected edge geodetic set of C_p and so W is not a minimal connected edge geodetic set of G, which is a contradiction. Thus $g^+_{1c}(G) = n + 1 = \frac{p}{2} + 1$.
Case 2. p is odd. Let $p = 2n+1$. Let $C_{2n+1} : v_1, v_2, v_3, \ldots, v_{2n+1}, v_1$ be the cycle of order $2n + 1$. Let $S = \{v_1, v_2, v_3, \ldots, v_{n+2}\}$. Then, as in Case 1 it is seen that S is a minimal connected edge geodetic set of C_p and $g^+_{1c}(C_p) = n + 2 = \lfloor \frac{p}{2} \rfloor + 2$. □

The next theorem follows from Theorem 1.4 and Proposition 2.3.

Theorem 2.13. *For a connected graph G, $g^+_{1c}(G) \geq 1 + diam\ G$.*

For every connected graph, rad $G \leq$ diam $G \leq 2$ rad G. Ostrand [6] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the upper connected edge geodetic number can also be prescribed.

In view of Theorem 2.13, we have the following realization result.

Theorem 2.14. *For positive integers r, d and $n \geq d + 1$ with $r \leq d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $g^+_{1c}(G) = n$.*

Proof. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_n$. Then by Corollary 2.9 (i), $g^+_{1c}(G) = n$. If $d = 2$, let $G = K_{1,n-1}$. Then by Corollary 2.9 (ii), $g^+_{1c}(G) = n$. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows:
Case 1. $r = d$. For $n = d + 1$, let $G = C_{2r}$. Then it is clear that $r = d$ and by Theorem 2.12, $g^+_{1c}(G) = d+1 = n$. Now, let $n \geq d+2$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$, be the cycle of order $2r$. Let G be the graph obtained from C_{2r} by adding the new vertices $x_1, x_2, \ldots, x_{n-r-1}$ and joining each x_i $(1 \leq i \leq n-r-1)$ with u_1 and u_2 of C_{2r}. The graph G is shown in Figure 2.3. It is easily verified that the eccentricity of each vertex of G is r so that rad G = diam $G = r$. Let $S = \{x_1, x_2, ..., x_{n-r-1}\}$. Then S is the set of all extreme vertices of G with $|S| = n-r-1$. It is clear that S is not a connected edge geodetic set of G. Let $T = S \cup \{u_1, u_2, u_3, ..., u_{r+1}\}$. Then T is a connected edge geodetic set of G. We claim that T is a minimal connected edge geodetic set of G. Assume, to the contrary, that T is not a minimal connected

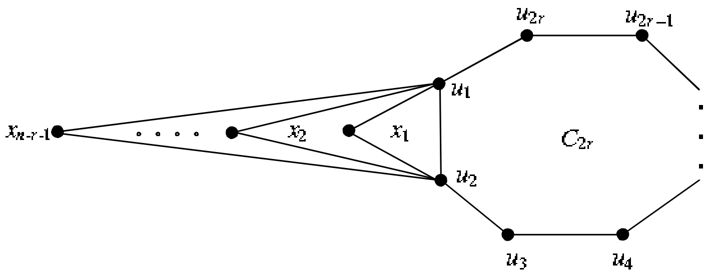

Figure 2.3: G

edge geodetic set of G. Then there is a proper subset M of T such that M is a connected edge geodetic set of G. Furthermore, it follows from Theorem 1.1 that every vertex of T that is not in M can only be u_i for some i.

Case 1a. $(T - M) \cap \{u_1, u_{r+1}\} \neq \phi$. Assume without loss of generality that $u_1 \in T - M$. Then it is clear that the edge $u_{r+1}u_{r+2}$ does not lie on any geodesic joining a pair of vertices of M and so M is not a connected edge geodetic set of G, which is a contradiction.

Case 1b. $(T - M) \cap \{u_1, u_{r+1}\} = \phi$. Then there exists a vertex $x \in (T - M) \cap \{u_2, u_3, \ldots, u_r\}$ and thus the subgraph induced by M is not connected and so M is not a connected edge geodetic set of G, which is a contradiction. Thus T is a minimal connected edge geodetic set of G and so $g^+_{1c}(G) \geq |T| = n$.

Now, we prove that $g^+_{1c}(G) = n$. Suppose that $g^+_{1c}(G) > n$. Let M' be a minimal connected edge geodetic set with $|M'| = m > n$. By Theorem 1.1, M' contains S and since $m > n$, M' contains at least $r+2$ vertices of C_{2r}. Since M' is a connected edge geodetic set of G, u_1 or u_2 must belong to M' and the subgraph induced by $M' - S$ is a path, say $u_2, u_3, ..., u_{m-n+r+2}$. It is clear that $T = S \cup \{u_2, u_3, ..., u_{r+2}\}$ is a connected edge geodetic set of G and so M' is not a minimal connected edge geodetic set of G. Thus $g^+_{1c}(G) = n$.

Case 2. Suppose $r < d \leq 2r$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d-r+1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Now, we add $n-d-1$ new vertices $w_1, w_2, \ldots, w_{n-d-1}$ to the graph H and join each vertex $w_i (1 \leq i \leq n-d-1)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 2.4. Then rad $G = r$ and diam $G = d$. Let $S = \{v_1, u_1, u_2, \ldots, u_{d-r-1}, u_{d-r}, w_1, w_2, \ldots, w_{n-d-1}\}$ be the set of all cut-vertices and extreme vertices of G. By Theorems 1.1 and 1.3, every connected edge geodetic set of G contains S. It is clear that S is not a connected edge geodetic set of G. Let $T = S \cup \{v_2, v_3, \ldots, v_{r+1}\}$. It is clear that T is a connected edge geodetic set of G and so $g_{1c}(G) \leq |T| = n$. Then by an argument similar to that given in the proof of Case 1 of this theorem, it can be proved that $g^+_{1c}(G) = n$. $\square$

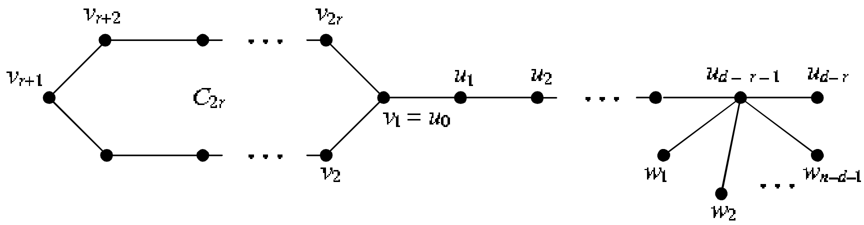

Figure 2.4: G

In view of Proposition 2.3, we have the following realization result.

Theorem 2.15. *For any positive integers $2 \le a < b \le c$, there exists a connected graph G such that $g_1(G) = a$, $g_{1c}(G) = b$ and $g^+_{1c}(G) = c$.*

Proof. If $2 \le a < b = c$, let G be any tree of order b with a end-vertices. Then it follows respectively from Theorem 1.3 and Corollary 2.9 (ii) that $g_1(G) = a, g_{1c}(G) = b$ and $g^+_{1c}(G) = c$. Let $2 \le a < b < c$. Now, we consider four cases.
Case 1. $a > 2$ and $b-a \ge 2$. Then $b-a+2 \ge 4$. Let $P_{b-a+2} : v_1, v_2, ..., v_{b-a+2}$ be a path of length $b-a+1$. Add $c-b+a-1$ new vertices $w_1, w_2, ..., w_{c-b+1}, u_1, u_2, \ldots, u_{a-2}$ to P_{b-a+2} and join $w_1, w_2, ..., w_{c-b+1}$ to both v_1 and v_3 and also join $u_1, u_2, ..., u_{a-2}$ to v_2, there by producing the graph G of Figure 2.5. Let $S = \{u_1, u_2, ..., u_{a-2}, v_{b-a+2}\}$ be the set of all extreme vertices of G. It is clear that S is not an edge geodetic set of G. Since $S \cup \{v_1\}$ is an edge geodetic set of G, it follows from Theorem 1.1 that $g_1(G) = a$. Let $S_1 = S \cup \{v_2, v_3, v_4, ..., v_{b-a+1}\}$. It is clear that S_1 is not a connected edge geodetic set of G. Since $S_1 \cup \{v_1\}$ is a connected edge geodetic set of G it follows from Theorems 1.1 and 1.3 that $g_{1c}(G) = b$.

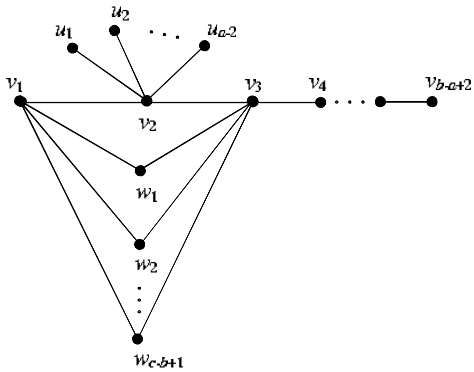

Figure 2.5: G

Let $S_2 = S_1 \cup \{w_1, w_2, ..., w_{c-b+1}\}$. It is clear that S_2 is a connected edge geodetic set of G. If S_2 is not a minimal connected edge geodetic set, then there is a proper subset T of S_2 such that T is a connected edge geodetic set of G. Let $v \in S_2$ and

$v \notin T$. By Theorems 1.1 and 1.3, it is clear $v = w_i$, for some $i = 1, 2, ..., c - b + 1$. Clearly, this w_i does not lie on a geodesic joining any pair of vertices of T and so T is not a connected edge geodetic set of G, which is a contradiction. Thus S_2 is a minimal connected edge geodetic set of G and so $g^+_{1c}(G) \geq c$. Since the order of the graph is $c + 1$, it follows that $g^+_{1c}(G) = c$.

Case 2. Let $a > 2$ and $b - a = 1$. Since $c > b$, we have $c - b + 1 \geq 2$. For the graph G given in Figure 2.6, we can prove as in case 1, $S = \{u_1, u_2, ..., u_{a-2}, v_1, v_3\}$ is a minimum edge geodetic set, $S_1 = S \cup \{v_2\}$ is a minimum connected edge geodetic set and $S_2 = V(G) - \{v_1\}$ is a minimal connected edge geodetic set of G so that $g_1(G) = a,\ g_{1c}(G) = b$ and $g^+_{1c}(G) = c$.

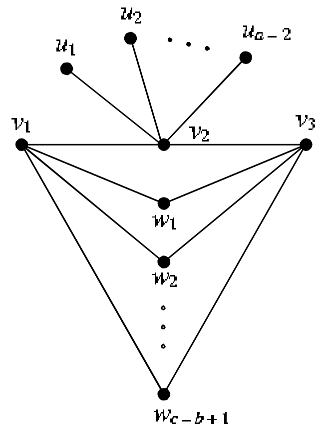

Figure 2.6: G

Case 3. Let $a = 2$ and $b - a = 1$. Then $b = 3$. For the graph G given in Figure 2.7, we can prove as in case 1, $S = \{v_1, v_3\}$ is a minimum edge geodetic set, $S_1 = \{v_1, v_2, v_3\}$ is a minimum connected edge geodetic set and $S_2 = V(G) - \{v_1\}$ is a minimal connected edge geodetic set so that $g_1(G) = a,\ g_{1c}(G) = b$ and $g^+_{1c}(G) = c$.

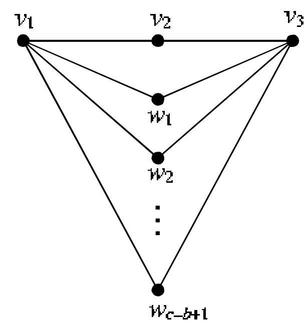

Figure 2.7: G

Case 4. Let $a = 2$ and $b - a \geq 2$. Then $b \geq 4$. For the graph G given in Figure 2.8, we can prove as in Case 1, $S = \{v_1, v_b\}$ is a minimum edge geodetic set, $S_1 = \{v_1, v_2, ..., v_b\}$ is a minimum connected edge geodetic set, $S_2 = V(G) - \{v_1\}$ is a minimal connected edge geodetic set so that $g_1(G) = a$, $g_{1c}(G) = b$ and $g^+_{1c}(a) = c$. □

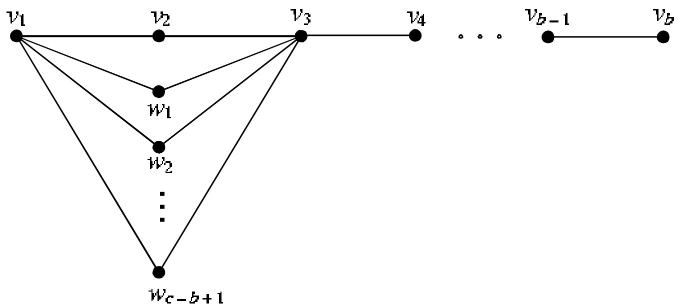

Figure 2.8: G

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