Filomat 26:2 (2012), 135–145
DOI (will be added later)

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A comment on some recent results concerning the Drazin inverse of an anti-triangular block matrix

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Abstract. In this note we give formulae for the Drazin inverse M^D of an anti-triangular special block matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ under some conditions expressed in terms of the individual blocks, which generalize some recent results given by Changjiang Bu [7, 8] and Chongguang Cao [10], etc.

1. Introduction

This research came up when we read some recent papers [7]-[10] which were concerned about calculating the Drazin inverses or group inverses of the anti-triangular special block matrices. The concept of the Drazin inverse plays an important role in various fields like Markov chains, singular differential and difference equations, iterative methods, etc. [1]-[6], [15]. Our purpose is to give representations for the Drazin inverse of the anti-triangular block matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ under some conditions expressed in terms of the individual blocks. Block matrices of this form arise in numerous applications, ranging from constrained optimization problems to the solution of differential equations [1], [2], [3], [13], [16], [17].

Let $P = P^2$ be an idempotent matrix. C. Cao in 2006 [10] gave the group inverse of every one of the seven matrices: $\begin{pmatrix} PP^* & P \\ P & 0 \end{pmatrix}$, $\begin{pmatrix} P & P \\ PP^* & 0 \end{pmatrix}$, $\begin{pmatrix} PP^* & PP^* \\ P & 0 \end{pmatrix}$, $\begin{pmatrix} P & P \\ P^* & 0 \end{pmatrix}$, $\begin{pmatrix} P & PP^* \\ PP^* & 0 \end{pmatrix}$, $\begin{pmatrix} P & PP^* \\ P^* & 0 \end{pmatrix}$ and $\begin{pmatrix} P^* & P \\ P & 0 \end{pmatrix}$. Recently, C. Bu, et al. in [7–9] has obtained the new representations for the group inverse of a 2×2 anti-triangular matrix $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$, where $A^2 = A$ in terms of the group inverse of AB. In the present paper we will find explicit expressions for the Drazin inverse of a 2×2 anti-triangular operator matrix M under other weaker constraints. Our results generalize some recent results given by Changjiang Bu [7, 8] and Chong Guang Cao [10], etc.

In this note, let A be an $n \times n$ complex matrix. We denote by $\mathcal{N}(A), \mathcal{R}(A)$ and rank(A) the null space, the range and the rank of matrix A, respectively. The Drazin inverse [2] of $A \in \mathbb{C}^{n\times n}$ is the unique complex

2010 *Mathematics Subject Classification*. 15A09; 47A05.
Keywords. Anti-triangular matrix, Drazin inverse, Group inverse, Block matrix.
Received: 4 April 2011; Accepted: 20 August 2012
Communicated by Dragana S. Cvetković-Ilić
Partial support was provided by a grant from the Ph.D. Programs Foundation (No.20094407120001)
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matrix $A^D \in \mathbb{C}^{n\times n}$ satisfying the relations

$$AA^D = A^D A,\ \ A^D AA^D = A^D,\ \ A^k AA^D = A^k \quad \text{for all } k \geq r, \tag{1}$$

where $r = \text{ind(A)}$, called the index of A, is the smallest nonnegative integer such that $\text{rank}(A^{r+1}) = \text{rank}(A^r)$. We will denote by $A^\pi = I - AA^D$ the projection on $\mathcal{N}(A^r)$ along $\mathcal{R}(A^r)$. In the case $\text{ind(A)} = 1$, A^D reduces to the group inverse of A, denoted by $A^\#$. In particular, A is nonsingular if and only if $\text{ind}(A) = 0$.

2. Key lemmas

In this section, we state some lemmas which will be used to prove our main results.

Lemma 2.1. (see [7, Lemma 2.5]) *Let $A, B \in \mathbb{C}^{n\times n}$ such that* $\text{rank}(A) = r$. *If $A^2 = A$ and* $\text{rank}(B) = \text{rank}(BAB)$, *then A and B can be written as*

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}$$

with respect to space decomposition $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$, where AB, BA and $B_1 \in \mathbb{C}^{r\times r}$ are group invertible, $X \in \mathbb{C}^{r\times(n-r)}$ and $Y \in \mathbb{C}^{(n-r)\times r}$.

The following lemma concerns the Drazin inverse of 2×2 block matrix.

Lemma 2.2. (see Lemma 2.2 and Corollary 2.3 in [14]) *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that D is nilpotent and* $\text{ind(D)} = \text{s}$. *If $BC = 0$ and $BD = 0$, then*

$$M^D = \begin{pmatrix} A^D & (A^D)^2 B \\ \sum\limits_{i=0}^{s-1} D^i C (A^D)^{i+2} & \sum\limits_{i=0}^{s-1} D^i C (A^D)^{i+3} B \end{pmatrix}.$$

Lemma 2.3. (see [11, Theorem 2.3]) Let $A, B \in \mathbb{C}^{n\times n}$ such that $AB = BA$. Then

(1) $(AB)^D = B^D A^D = A^D B^D$.

(2) $AB^D = B^D A$ and $A^D B = BA^D$.

(3) $(AB)^\pi = B^\pi$ when A is invertible.

Lemma 2.4. *Let $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ such that A is nilpotent and* $\text{ind(A)} = \text{s}$. *If $BA = 0$, then M is nilpotent with* $\text{ind(M)} \leq \text{s} + 1$.

Proof. Note that, if $BA = 0$, then $M^{s+1} = \begin{pmatrix} A^{s+1} + A^s B & A^s \\ 0 & 0 \end{pmatrix} = 0$.
□

Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where $A \in \mathbb{C}^{d\times d}$, $B \in \mathbb{C}^{d\times(n-d)}$ and $C \in \mathbb{C}^{(n-d)\times d}$. N. Castro-González and E. Dopazo (see [3, Theorem 4.1]) had proved that, if $CA^D A = C$ and $A^D BC = BCA^D$, then (see [3], pp.267)

$$M^D = \begin{pmatrix} (A^D)^2\left[W_1 + (A^D)^2 BCW_2\right](BC)^\pi A & \left[(BC)^D + (A^D)^2 W_1 (BC)^\pi\right] B \\ C\left[(BC)^D + (A^D)^2 W_1 (BC)^\pi\right] & C\left[-A((BC)^D)^2 + (A^D)^3 W_2 (BC)^\pi\right] B \end{pmatrix}, \tag{2}$$

where

$$r = \mathrm{ind}\left[(\mathrm{A^D})^2\mathrm{BC}\right], \quad \mathrm{W}_1 = \sum_{\mathrm{j=0}}^{\mathrm{r-1}}(-1)^{\mathrm{j}}\mathrm{C}(2\mathrm{j}+1,\mathrm{j})(\mathrm{A^D})^{2\mathrm{j}}(\mathrm{BC})^{\mathrm{j}}, \quad \mathrm{W}_2 = \sum_{\mathrm{j=0}}^{\mathrm{r-1}}(-1)^{\mathrm{j}}\mathrm{C}(2\mathrm{j}+2,\mathrm{j})(\mathrm{A^D})^{2\mathrm{j}}(\mathrm{BC})^{\mathrm{j}}.$$

As a directly application of [3, Theorem 4.1]) and Lemma 2.3, we get the following result.

Lemma 2.5. *Let* $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ *such that* A *is nonsingular and* ind(B) = r. *If* $BA = AB$, *then*

$$M^D = \begin{pmatrix} W_1B^\pi + W_2BB^\pi & B^D + W_1B^\pi \\ \left[BB^D + W_1BB^\pi\right]A^{-1} & -B^D + W_2BB^\pi \end{pmatrix},$$

where

$$W_1 = \sum_{j=0}^{r-1}(-1)^jC(2j+1,j)A^{-j-1}B^j \quad \text{and} \quad W_2 = \sum_{j=0}^{r-1}(-1)^jC(2j+2,j)A^{-j-2}B^j.$$

In Lemma 2.5, if $A = I$, then

$$\begin{pmatrix} I & I \\ B & 0 \end{pmatrix}^D = \begin{pmatrix} Y_1B^\pi & B^D + Y_2B^\pi \\ BB^D + Y_2BB^\pi & -B^D + (Y_1 - Y_2)B^\pi \end{pmatrix},$$

where $Y_2 = W_1 = \sum_{j=0}^{r-1}(-1)^jC(2j+1,j)B^j$ and $Y_1B^\pi = Y_2B^\pi + W_2BB^\pi = \sum_{j=0}^{r-1}(-1)^jC(2j,j)B^jB^\pi$. This result had been given by N. Castro-González and E. Dopazoin in their celebrated paper [3, Theorem 3.3].

3. Main results

Our first purpose is to obtain a representation for M^D of the matrix $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ under some conditions, where A, B are $n \times n$ matrices. Throughout our development, we will be concerned with the anti-upper-triangular matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. However, the results we obtain will have an analogue for anti-lower-triangular matrix $M = \begin{pmatrix} 0 & A \\ C & B \end{pmatrix}$. The following result generalizes the recent result given by Changjiang Bu, et al (see [7, Theorem 3.1]).

Theorem 3.1. *Let* $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ *and* $\widetilde{B} = (I - A^\pi)B(I - A^\pi)$ *with* ind(A) = s *and* ind($\widetilde{\mathrm{B}}$) = r. *If*

$$BAA^\pi = 0 \quad \text{and} \quad (I - A^\pi)(BA - AB)(I - A^\pi) = 0,$$

then

$$M^D = \left[R + \sum_{i=0}^{s}\begin{pmatrix} AA^\pi & AA^\pi \\ A^\pi BA^\pi & 0 \end{pmatrix}^i \begin{pmatrix} 0 & 0 \\ A^\pi B(I - A^\pi) & 0 \end{pmatrix} R^{i+2}\right] \times \left[I + R\begin{pmatrix} 0 & 0 \\ (I - A^\pi)BA^\pi & 0 \end{pmatrix}\right], \tag{3}$$

where

$$R = \begin{pmatrix} \Gamma_1\widetilde{B}^\pi + \Gamma_2\widetilde{B}\widetilde{B}^\pi & \widetilde{B}^D + \Gamma_1\widetilde{B}^\pi \\ \left[\widetilde{B}\widetilde{B}^D + \Gamma_1\widetilde{B}\widetilde{B}^\pi\right]A^D & -\widetilde{B}^D + \Gamma_2\widetilde{B}\widetilde{B}^\pi \end{pmatrix},$$

$$\Gamma_1 = \sum_{j=0}^{r-1}(-1)^jC(2j+1,j)(A^D)^{j+1}\widetilde{B}^j, \qquad \Gamma_2 = \sum_{j=0}^{r-1}(-1)^jC(2j+2,j)(A^D)^{j+2}\widetilde{B}^j. \tag{4}$$

Proof. Let $X_1 = \mathcal{N}(A^\pi)$ and $X_2 = \mathcal{R}(A^\pi)$. Then $X = X_1 \oplus X_2$. Since A is ind$(A) = s$, A has the form

$$A = A_1 \oplus A_2 \text{ with } A_1 \text{ nonsingular}, \quad A_2^s = 0 \quad \text{and} \quad A^D = A_1^{-1} \oplus 0. \tag{5}$$

Using the decomposition $X \oplus X = X_1 \oplus X_2 \oplus X_1 \oplus X_2$, we have

$$M = \begin{pmatrix} A_1 & 0 & A_1 & 0 \\ 0 & A_2 & 0 & A_2 \\ B_1 & B_3 & 0 & 0 \\ B_4 & B_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_1 \\ X_2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_1 \\ X_2 \end{pmatrix}. \tag{6}$$

Define $I_0 = I \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \oplus I$. It is clear that I_0, as a matrix from $X_1 \oplus X_2 \oplus X_1 \oplus X_2$ onto $X_1 \oplus X_1 \oplus X_2 \oplus X_2$, is nonsingular with $I_0 = I_0^* = I_0^{-1}$. Hence

$$M^D = \left[\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_1 & A_1 & 0 & 0 \\ B_1 & 0 & B_3 & 0 \\ 0 & 0 & A_2 & A_2 \\ B_4 & 0 & B_2 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \right]^D = I_0 \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^D I_0, \tag{7}$$

where

$$A_0 = \begin{pmatrix} A_1 & A_1 \\ B_1 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 0 \\ B_3 & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 \\ B_4 & 0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} A_2 & A_2 \\ B_2 & 0 \end{pmatrix}. \tag{8}$$

If $(I - A^\pi)(BA - AB)(I - A^\pi) = 0$, using the representations in (5) and (6), we get A_1 is nonsingular and $A_1 B_1 = B_1 A_1$. Since $\mathrm{ind}[(I - A^\pi)B(I - A^\pi)] = \mathrm{ind}[B_1] = r$, by Lemma 2.5, we get

$$\begin{aligned}
R &=: I_0 \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}^D I_0 \\
&= I_0 \begin{pmatrix} W_1 B_1^\pi + W_2 B_1 B_1^\pi & B_1^D + W_1 B_1^\pi & 0 & 0 \\ \left[B_1 B_1^D + W_1 B_1 B_1^\pi\right] A_1^{-1} & -B_1^D + W_2 B_1 B_1^\pi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} I_0 \\
&= \begin{pmatrix} W_1 B_1^\pi + W_2 B_1 B_1^\pi & 0 & B_1^D + W_1 B_1^\pi & 0 \\ 0 & 0 & 0 & 0 \\ \left[B_1 B_1^D + W_1 B_1 B_1^\pi\right] A_1^{-1} & 0 & -B_1^D + W_2 B_1 B_1^\pi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \Gamma_1 \widetilde{B}^\pi + \Gamma_2 \widetilde{B}\widetilde{B}^\pi & \widetilde{B}^D + \Gamma_1 \widetilde{B}^\pi \\ \left[\widetilde{B}\widetilde{B}^D + \Gamma_1 \widetilde{B}\widetilde{B}^\pi\right] A^D & -\widetilde{B}^D + \Gamma_2 \widetilde{B}\widetilde{B}^\pi \end{pmatrix},
\end{aligned}$$

where

$$W_1 = \sum_{j=0}^{r-1} (-1)^j C(2j+1, j) A_1^{-j-1} B_1^j, \qquad W_2 = \sum_{j=0}^{r-1} (-1)^j C(2j+2, j) A_1^{-j-2} B_1^j,$$

$$\Gamma_1 = \sum_{j=0}^{r-1} (-1)^j C(2j+1, j) (A^D)^{j+1} \widetilde{B}^j, \qquad \Gamma_2 = \sum_{j=0}^{r-1} (-1)^j C(2j+2, j) (A^D)^{j+2} \widetilde{B}^j.$$

Since $BAA^{\pi} = 0$, we get $B_3A_2 = 0$ and $B_2A_2 = 0$. By Lemma 2.4, we get D_0 is nilpotent with $\mathrm{ind}(D_0) \le s+1$. Note that $B_3A_2 = 0$ implies that $B_0C_0 = 0$ and $B_0D_0 = 0$. By Lemma 2.2, we obtain

$$\begin{aligned} M^D &= I_0 \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^D I_0 = I_0 \begin{pmatrix} A_0^D & (A_0^D)^2 B_0 \\ \sum_{i=0}^{s} D_0^i C_0 (A_0^D)^{i+2} & \sum_{i=0}^{s} D_0^i C_0 (A_0^D)^{i+3} B_0 \end{pmatrix} I_0 \\ &= I_0 \left[\begin{pmatrix} A_0^D & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=0}^{s} \begin{pmatrix} 0 & 0 \\ 0 & D_0^i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix} \begin{pmatrix} (A_0^D)^{i+2} & 0 \\ 0 & 0 \end{pmatrix} \right] \times \left[I + \begin{pmatrix} A_0^D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} \right] I_0 \qquad (9) \\ &= \left[R + \sum_{i=0}^{s} \begin{pmatrix} AA^{\pi} & AA^{\pi} \\ A^{\pi}BA^{\pi} & 0 \end{pmatrix}^i \begin{pmatrix} 0 & 0 \\ A^{\pi}B(I-A^{\pi}) & 0 \end{pmatrix} R^{i+2} \right] \times \left[I + R \begin{pmatrix} 0 & 0 \\ (I-A^{\pi})BA^{\pi} & 0 \end{pmatrix} \right]. \end{aligned}$$

□

We remark that, from the above theorem we get the following corollaries.

Corollary 3.2. *Let* $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$.

(i) *If* $AB = BA$, $\mathrm{ind}(A) = 1$ *and* $\mathrm{ind}(B) = r$, *then*

$$M^D = \begin{pmatrix} \Gamma_1 B^{\pi} + \Gamma_2 BB^{\pi} & (I - A^{\pi})B^D + \Gamma_1 B^{\pi} \\ \left[BB^D + \Gamma_1 BB^{\pi} \right] A^D & -(I-A^{\pi})B^D + \Gamma_2 BB^{\pi} \end{pmatrix},$$

where

$$\Gamma_1 = \sum_{j=0}^{r-1} (-1)^j C(2j+1, j)(A^{\#})^{j+1} B^j, \qquad \Gamma_2 = \sum_{j=0}^{r-1} (-1)^j C(2j+2, j)(A^{\#})^{j+2} B^j.$$

(ii) *If* A, B *are group invertible and* $AB = BA$, *then*

$$M^D = \begin{pmatrix} A^{\#}B^{\pi} & (I-A^{\pi})B^{\#} + A^{\#}B^{\pi} \\ [I - B^{\pi}]A^{\#} & -(I-A^{\pi})B^{\#} \end{pmatrix}.$$

In addition, if $A^{\pi}B = 0$, *then* $A^{\pi}B^{\#} = 0$, M^D *becomes the group inverse and*

$$M^{\#} = \begin{pmatrix} A^{\#}B^{\pi} & B^{\#} + A^{\#}B^{\pi} \\ [I - B^{\pi}]A^{\#} & -B^{\#} \end{pmatrix}.$$

(iii) *If* A, B *are invertible, then*

$$M^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & -B^{-1} \end{pmatrix}.$$

Corollary 3.3. *Let* $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$, *where* $A, B \in \mathbb{C}^{n\times n}$, $A = A^2$ *and* $\mathrm{ind}(ABA) = r$. *Then*

(i) (see [9, Theorem 3.2])

$$M^D = \left[R + \begin{pmatrix} 0 & 0 \\ (I-A)BA & 0 \end{pmatrix} R^2 \right] \left[I + R \begin{pmatrix} 0 & 0 \\ AB(I-A) & 0 \end{pmatrix} \right], \qquad (10)$$

where

$$R = \begin{pmatrix} X + Y & (AB)^D A + X \\ \left[(AB)^D + X \right] ABA & -(AB)^D A + Y \end{pmatrix},$$

$$X = \sum_{j=0}^{r-1} (-1)^j C(2j+1, j)(AB)^{\pi}(AB)^j A, \qquad Y = \sum_{j=0}^{r-1} (-1)^j C(2j+2, j)(AB)^{\pi}(AB)^{j+1} A.$$

(ii) (see [7, Theorem 3.1]) $M^{\#}$ exists if and only if rank(B) = rank(BAB) and

$$M^{\#} = \begin{pmatrix} A-(AB)^{\#}+(AB)^{\#}A-(AB)^{\#}ABA & A+(AB)^{\#}A+(AB)^{\#}ABA \\ (BA)^{\#}B-(BA)^{\#}(AB)^{\#}AB-(BA)^{\#} & -(BA)^{\#} \end{pmatrix}. \tag{11}$$

Proof. (i) If $A = A^2$, we have ind(A) = 1, $A = A^D$, $A^{\pi} = I - A$,

$$\widetilde{B}^D = [(I-A^{\pi})B(I-A^{\pi})]^D = (ABA)^D = AB[(AAB)^D]^2A = (AB)^DA$$

and

$$\widetilde{B}^j = (ABA)^j = (AB)^jA = A(BA)^j.$$

So

$$\widetilde{B}^{\pi} = (ABA)^{\pi} = I-(ABA)^D(ABA) = I-(AB)^DABA = I-A+(AB)^{\pi}A = I-A+A(BA)^{\pi}.$$

Hence, Γ_1 and Γ_2 in (3.2) reduce as

$$\Gamma_1 = \sum_{j=0}^{r-1}(-1)^jC(2j+1,j)(A^D)^{j+1}\widetilde{B}^j = \sum_{j=0}^{r-1}(-1)^jC(2j+1,j)(AB)^jA,$$

$$\Gamma_2 = \sum_{j=0}^{r-1}(-1)^jC(2j+2,j)(A^D)^{j+2}\widetilde{B}^j = \sum_{j=0}^{r-1}(-1)^jC(2j+2,j)(AB)^jA.$$

Let

$$X = \Gamma_1\widetilde{B}^{\pi} = \sum_{j=0}^{r-1}(-1)^jC(2j+1,j)(AB)^{\pi}(AB)^jA, \qquad Y = \Gamma_2\widetilde{B}\widetilde{B}^{\pi} = \sum_{j=0}^{r-1}(-1)^jC(2j+2,j)(AB)^{\pi}(AB)^{j+1}A.$$

Then R in (3.2) reduces as

$$R = \begin{pmatrix} \Gamma_1\widetilde{B}^{\pi}+\Gamma_2\widetilde{B}\widetilde{B}^{\pi} & \widetilde{B}^D+\Gamma_1\widetilde{B}^{\pi} \\ \left[\widetilde{B}\widetilde{B}^D+\Gamma_1\widetilde{B}\widetilde{B}^{\pi}\right]A^D & -\widetilde{B}^D+\Gamma_2\widetilde{B}\widetilde{B}^{\pi} \end{pmatrix} = \begin{pmatrix} X+Y & (AB)^DA+X \\ \left[(AB)^D+X\right]ABA & -(AB)^DA+Y \end{pmatrix}.$$

By Theorem 3.1, we get

$$\begin{aligned} M^D &= \left[R+\sum_{i=0}^{1}\begin{pmatrix} 0 & 0 \\ (I-A)B(I-A) & 0 \end{pmatrix}^i\begin{pmatrix} 0 & 0 \\ (I-A)BA & 0 \end{pmatrix}R^{i+2}\right]\times\left[I+R\begin{pmatrix} 0 & 0 \\ AB(I-A) & 0 \end{pmatrix}\right] \\ &= \left[R+\begin{pmatrix} 0 & 0 \\ (I-A)BA & 0 \end{pmatrix}R^2\right]\left[I+R\begin{pmatrix} 0 & 0 \\ AB(I-A) & 0 \end{pmatrix}\right]. \end{aligned}$$

(ii) See Theorem 3.1 in [7] for the proof that $M^{\#}$ exists if and only if rank(B) = rank(BAB). By Lemma 2.1, we have ind(ABA) ≤ 1, AB and BA are group invertible. So, by item (i), we get $X = (AB)^{\pi}A$, $Y = 0$,

$$R = \begin{pmatrix} (AB)^{\pi}A & (AB)^{\#}A+(AB)^{\pi}A \\ (AB)^{\#}ABA & (AB)^{\#}A-(AB)^{\#}A \end{pmatrix}$$

and

$$R^2 = \begin{pmatrix} (AB)^{\pi}A+(AB)^{\#}A & (AB)^{\pi}A-[(AB)^{\#}]^2A \\ -(AB)^{\#}A & (AB)^{\#}A+[(AB)^{\#}]^2A \end{pmatrix}.$$

Thus, collecting the above computations in the expression (10) for M^D, we get the statement of (11). □

Note that

$$\sum_{j=0}^{r-1}(-1)^j C(2j,j)B^jB^\pi = \sum_{j=0}^{r-1}(-1)^j C(2j+1,j)B^jB^\pi + \sum_{j=0}^{r-1}(-1)^j C(2j+2,j)B^{j+1}B^\pi.$$

In Corollary 3.3, if we set $A = I$ and $Z = \sum_{j=0}^{r-1}(-1)^j C(2j,j)B^jB^\pi$, then we get $Y = Z - X$ and Corollary 3.3 (resp. Theorem 3.1) reduces as the following result which had been given in [3].

Corollary 3.4. ([3, Theorem 3.3]) Let $M = \begin{pmatrix} I & I \\ B & 0 \end{pmatrix}$, where $B \in \mathbb{C}^{n\times n}$ and ind(B) = r. Then

$$M^D = \begin{pmatrix} Z & B^D + X \\ B^DB + XB & -B^D + Z - X \end{pmatrix},$$

where $X = \sum_{j=0}^{r-1}(-1)^j C(2j+1,j)B^jB^\pi, \quad Z = \sum_{j=0}^{r-1}(-1)^j C(2j,j)B^jB^\pi.$

Our next purpose is to obtain a representation for the Drazin inverse of block antitriangular matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where $A, C \in \mathbb{C}^{n\times n}$, which in some different ways generalizes recent results given in [10, 14]. We start introducing a different method to give matrix block representation. Let $S = -CA^DB$, ind$(A) = m$ and ind$(S) = n$. In (5) and (6), if we set

$$X_1 = \mathcal{N}(A^\pi), \quad X_2 = \mathcal{R}(A^\pi), \quad Y_1 = \mathcal{N}(S^\pi) \quad \text{and} \quad Y_2 = \mathcal{R}(S^\pi).$$

Then $X \oplus Y = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$. In this case, A and S have the forms

$$\begin{aligned} &A = A_1 \oplus A_2 \text{ with } A_1 \text{ nonsingular}, \quad A_2^m = 0 \quad \text{and} \quad A^D = A_1^{-1} \oplus 0, \\ &S = S_1 \oplus S_2 \text{ with } S_1 \text{ nonsingular}, \quad S_2^n = 0 \quad \text{and} \quad S^D = S_1^{-1} \oplus 0. \end{aligned} \tag{12}$$

Using the decomposition $X \oplus Y = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$, we have

$$M = \begin{pmatrix} A_1 & 0 & B_1 & B_3 \\ 0 & A_2 & B_4 & B_2 \\ C_1 & C_3 & 0 & 0 \\ C_4 & C_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix}. \tag{13}$$

Note that the generalized Schur complement

$$S = S_1 \oplus S_2 = -CA^DB = -\begin{pmatrix} C_1 & C_3 \\ C_4 & C_2 \end{pmatrix}\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix} = \begin{pmatrix} -C_1A_1^{-1}B_1 & -C_1A_1^{-1}B_3 \\ -C_4A_1^{-1}B_1 & -C_4A_1^{-1}B_3 \end{pmatrix}.$$

Comparing the two sides of the above equation, we have

$$S_1 = -C_1A_1^{-1}B_1, \quad S_2 = -C_4A_1^{-1}B_3, \quad C_1A_1^{-1}B_3 = 0 \text{ and } C_4A_1^{-1}B_1 = 0.$$

In this case, $I_0 = I \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \oplus I$ as a matrix from $X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$ onto $X_1 \oplus Y_1 \oplus X_2 \oplus Y_2$ is nonsingular with $I_0 = I_0^* = I_0^{-1}$. Hence

$$M^D = I_0\begin{pmatrix} A_1 & B_1 & 0 & B_3 \\ C_1 & 0 & C_3 & 0 \\ 0 & B_4 & A_2 & B_2 \\ C_4 & 0 & C_2 & 0 \end{pmatrix}^D I_0 := I_0\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^D I_0, \tag{14}$$

where

$$A_0 = \begin{pmatrix} A_1 & B_1 \\ C_1 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & B_3 \\ C_3 & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & B_4 \\ C_4 & 0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} A_2 & B_2 \\ C_2 & 0 \end{pmatrix}. \tag{15}$$

Since the Schur complement of A_1 in A_0 is $-C_1A_1^{-1}B_1 = S_1$ and S_1 is nonsingular, it follows that A_0 is nonsingular with

$$A_0^{-1} = \begin{pmatrix} A_1^{-1} + A_1^{-1}B_1S_1^{-1}C_1A_1^{-1} & -A_1^{-1}B_1S_1^{-1} \\ -S_1^{-1}C_1A_1^{-1} & S_1^{-1} \end{pmatrix}. \tag{16}$$

Let $R = I_0 \begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix} I_0$. Using the rearrangement effect of I_0, we get

$$R = \begin{pmatrix} A^D + A^DBS^DCA^D & -A^DBS^D \\ -S^DCA^D & S^D \end{pmatrix}. \tag{17}$$

The expression (17) is called the generalized-Banachiewicz-Schur form of the matrix M and can be found in some recent papers [14].

Now, we are in position to prove the following theorem which provides expressions for M^D.

Theorem 3.5. *Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $S = -CA^DB$ with $ind(A) = m$. If*

$$(I - S^\pi)CA^\pi B = 0, \quad (I - S^\pi)CA^\pi A = 0, \quad (I - A^\pi)BS^\pi C = 0, \quad BS^\pi CA^\pi = 0, \tag{18}$$

then

$$M^D = \left[R + \sum_{i=0}^{m+1} \begin{pmatrix} AA^\pi & A^\pi BS^\pi \\ S^\pi CA^\pi & 0 \end{pmatrix}^i \begin{pmatrix} 0 & A^\pi B(I - S^\pi) \\ S^\pi C(I - A^\pi) & 0 \end{pmatrix} R^{i+2}\right] \times \left[I + R\begin{pmatrix} 0 & (I - A^\pi)BS^\pi \\ (I - S^\pi)CA^\pi & 0 \end{pmatrix}\right].$$

where R is defined as in (17).

Proof. Let A_0, B_0, C_0 and D_0 be defined by (15). Similar to the proof of Theorem 3.1, it is trivial to check that the conditions in (18) imply that $B_0C_0 = 0$ and $B_0D_0 = 0$. Note that

$$\begin{pmatrix} A_2 & B_2 \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} A_2^k & A_2^{k-1}B_2 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{for} \quad k \geq m + 1.$$

The condition $BS^\pi CA^\pi = 0$ implies that $B_2C_2 = 0$ and $\begin{pmatrix} A_2 & B_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix} = 0$. So

$$\begin{aligned} D_0^{m+2} &= \begin{pmatrix} A_2 & B_2 \\ C_2 & 0 \end{pmatrix}^{m+2} = \left[\begin{pmatrix} A_2 & B_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}\right]^{m+2} \\ &= \begin{pmatrix} A_2 & B_2 \\ 0 & 0 \end{pmatrix}^{m+2} + \begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ 0 & 0 \end{pmatrix}^{m+1} = 0. \end{aligned} \tag{19}$$

D_0 is nilpotent and $\text{ind}(D_0) \le m+2$. By Lemma 2.2 and the proof in (9), we obtain

$$\begin{aligned} M^D &= I_0 \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^D I_0 = I_0 \begin{pmatrix} A_0^D & (A_0^D)^2 B_0 \\ \sum_{i=0}^{m+1} D_0^i C_0 (A_0^D)^{i+2} & \sum_{i=0}^{m+1} D_0^i C_0 (A_0^D)^{i+3} B_0 \end{pmatrix} I_0 \\ &= I_0 \left[\begin{pmatrix} A_0^D & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=0}^{m+1} \begin{pmatrix} 0 & 0 \\ 0 & D_0^i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix} \begin{pmatrix} (A_0^D)^{i+2} & 0 \\ 0 & 0 \end{pmatrix} \right] \times \left[I + \begin{pmatrix} A_0^D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} \right] I_0 \\ &= \left[R + \sum_{i=0}^{m+1} \begin{pmatrix} AA^\pi & A^\pi B S^\pi \\ S^\pi C A^\pi & 0 \end{pmatrix}^i \begin{pmatrix} 0 & A^\pi B(I-S^\pi) \\ S^\pi C(I-A^\pi) & 0 \end{pmatrix} R^{i+2} \right] \\ &\quad \times \left[I + R \begin{pmatrix} 0 & (I-A^\pi) B S^\pi \\ (I-S^\pi) C A^\pi & 0 \end{pmatrix} \right]. \end{aligned}$$

□

We remark that our result has generalized some results in the literature. In [10], Chong Guang Cao has given the group inverse of $M = \begin{pmatrix} PP^* & P \\ P & 0 \end{pmatrix}$, where P is an idempotent. Note that

$$(PP^*)^D = (PP^*)^\# = (PP^*)^+ \quad \text{and} \quad PP^*(PP^*)^D P = PP^*(PP^*)^+ P = P.$$

If $A = PP^*$, $B = C = P$ in Theorem 3.5, then

$$S = -CA^D B = -P(PP^*)^D P = -(PP^*)^D P, \quad S^D = -PP^* P, \quad S^\pi = I - P.$$

It follows that

$$A^\pi B = 0, \quad A^\pi A = 0, \quad BS^\pi = 0, \quad S^\pi C = 0$$

and R in (17) reduces as

$$R = \begin{pmatrix} A^D + A^D B S^D C A^D & -A^D B S^D \\ -S^D C A^D & S^D \end{pmatrix} = \begin{pmatrix} 0 & P \\ PP^*(PP^*)^D & -PP^* P \end{pmatrix}.$$

By a direct computation we get the following result.

Corollary 3.6. [10, Theorem 2.1] *Let P be an idempotent matrix and $M = \begin{pmatrix} PP^* & P \\ P & 0 \end{pmatrix}$. Then*

$$M^\# = R\left[I + R \begin{pmatrix} 0 & 0 \\ P[I - PP^*(PP^*)^D] & 0 \end{pmatrix} \right] = \begin{pmatrix} PP^*(I-P) & P \\ (PP^*)^2(P-I) + P & -PP^* P \end{pmatrix}.$$

In Corollay 3.6, the reason that M^D is replaced by $M^\#$ is M satisfies the relation $MM^D M = M$. Similar to Corollary 3.6, if M is the matrix from the set

$$\left\{ \begin{pmatrix} P & P \\ PP^* & 0 \end{pmatrix}, \begin{pmatrix} PP^* & PP^* \\ P & 0 \end{pmatrix}, \begin{pmatrix} P & P \\ P^* & 0 \end{pmatrix}, \begin{pmatrix} P & PP^* \\ PP^* & 0 \end{pmatrix}, \begin{pmatrix} P & PP^* \\ P^* & 0 \end{pmatrix} \right\},$$

then M satisfies Theorem 3.5. Hence, Theorem 2.1–Theorem 2.6 in [10] are all the special cases of our Theorem 3.5.

If A in Theorem 3.5 is nonsingular and $A^{-1}BC$ is group invertible, then $A^\pi = 0$ and $\text{ind}(A) = 0$ and

$$\begin{aligned} 0 &= BC(A^{-1}BC)^\pi = BC - BC(A^{-1}BC)^D A^{-1}BC = BC - BCA^{-1}BC[(A^{-1}BC)^D]^2 A^{-1}BC \\ &= B\left[I - CA^{-1}BC[(A^{-1}BC)^D]^2 A^{-1}B\right]C = B\left[I - CA^{-1}B(CA^{-1}B)^D\right]C = BS^\pi C. \end{aligned}$$

From the above computations, we get the conditions in (18) hold and Theorem 3.5 reduces as the following:

Corollary 3.7. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, *A be nonsingular,* $S = -CA^{-1}B$ *such that* $A^{-1}BC$ *is group invertible, then*

$$M^D = \left[R + \begin{pmatrix} 0 & 0 \\ S^\pi C & 0 \end{pmatrix} R^2\right] \times \left[I + R\begin{pmatrix} 0 & BS^\pi \\ 0 & 0 \end{pmatrix}\right].$$

Finally, we derive from Theorem 3.5 some particular representations of A^D under certain additional conditions.

Corollary 3.8. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ *and* $S = -CA^DB$ *with* $ind(A) = m$. *Let R be defined as in (17).*

(i) *If* $C(I - AA^D) = 0$ *and the generalized Schur complement* $S = -CA^DB$ *is nonsingular, then*

$$M^D = R + \sum_{i=0}^{m+1} \begin{pmatrix} AA^\pi & 0 \\ 0 & 0 \end{pmatrix}^i \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} R^{i+2}.$$

(ii) (see [14, Theorem 1.1]) *If* $C(I - AA^D) = 0$, $(I - AA^D)B = 0$ *and the generalized Schur complement* $S = -CA^DB$ *is nonsingular, then*

$$M^D = \begin{pmatrix} A^D + A^DBS^{-1}CA^D & -A^DBS^{-1} \\ -S^{-1}CA^D & S^{-1} \end{pmatrix}.$$

(iii) (see [14, Theorem 3.1 or Corollary 3.2]) *If* $C(I - AA^D)B = 0$, $C(I - AA^D)A = 0$ *and the generalized Schur complement* $S = -CA^DB$ *is nonsingular, then*

$$M^D = \left[I + \sum_{i=0}^{m-1} \begin{pmatrix} 0 & A^iA^\pi B \\ 0 & 0 \end{pmatrix} R^{i+1}\right] R \left[I + R\begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}\right].$$

In Corollary 3.8(iii), if $CA^\pi = 0$, $A^\pi B = 0$ and the generalized Schur complement $S = -CA^DB$ is nonsingular,, then $M^D = R$, which is famous Banachiewicz-Schur formula.

4. Acknowledgments

The authors thank Prof. Dragana S. Cvetković-Ilić and the referee for their very useful and detailed comments which greatly improve the presentation.

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