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BCC-algebras with pseudo-valuations

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Abstract. The notion of pseudo-valuations (valuations) on a BCC-algebra is introduced by using the Buşneag's model ([1–3]), and a pseudo-metric is induced by a pseudo-valuation on BCC-algebras. Conditions for a real-valued function to be an BCK-pseudo-valuation are provided. The fact that the binary operation in BCC-algebras is uniformly continuous is provided based on the notion of (pseudo) valuation.

1. Introduction

In 1966, Y. Imai and K. Iséki (cf. [8]) defined a class of algebras of type $(2, 0)$ called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (cf. [8]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [12]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori (cf. [10]) introduced a notion of BCC-algebras, and W. A. Dudek (cf. [4, 5]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [7], W. A. Dudek and X. H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. Buşneag [2] defined a pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Also, Buşneag [3] provided several theorems on extensions of pseudo-valuations. Buşneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras ([3])).

In this paper, using the Buşneag's model, we introduce the notion of (BCK, BCC, strong BCC)-pseudo-valuations (valuations) on BCC-algebras, and we induce a pseudo-metric by using a BCK-pseudo-valuation on BCC-algebras. We provide conditions for a real-valued function on a BCC-algebra X to be a BCK-pseudo-pseudo-valuation on X. Based on the notion of (pseudo) valuation, we show that the binary operation $*$ in BCC-algebras is uniformly continuous.

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2. Preliminaries

Recall that a *BCC-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms:

(C1) $((x * y) * (z * y)) * (x * z) = 0$,

(C2) $0 * x = 0$,

(C3) $x * 0 = x$,

(C4) $x * y = 0$ and $y * x = 0$ imply $x = y$

for every $x, y, z \in X$. For any BCC-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X. In a BCC-algebra X, the following holds:

(a1) $(\forall x \in X)\ (x * x = 0)$,

(a2) $(\forall x, y \in X)\ (x * y \leq x)$,

(a3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.

A subset I of a BCC-algebra X is called a *BCK-ideal* if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

A subset I of a BCC-algebra X is called a *BCC-ideal* if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x, z \in X)\ (\forall y \in I)\ ((x * y) * z \in I \Rightarrow x * z \in I)$.

3. Pseudo-valuations on BCC-algebras

Definition 3.1. A real-valued function φ on a BCC-algebra X is called a *weak pseudo-valuation* on X if it satisfies the following condition:

$$(\forall x, y \in X)\ (\varphi(x * y) \leq \varphi(x) + \varphi(y)). \tag{1}$$

Definition 3.2. A real-valued function φ on a BCC-algebra X is called a *BCK-pseudo-valuation* on X if it satisfies the following condition:

$$\varphi(0) = 0, \tag{2}$$
$$(\forall x, y \in X)\ (\varphi(x * y) \geq \varphi(x) - \varphi(y)). \tag{3}$$

Example 3.3. Let $X := \{0, 1, 2, 3, 4\}$ be a BCC-algebra ([7]), which is not a BCK-algebra, with $*$-operation given by Table 1. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 4 & 5 \end{pmatrix}.$$

It is easy to check that φ is both a weak pseudo-valuation and a BCK-pseudo-valuation on X.

Proposition 3.4. *For a weak pseudo-valuation φ on a BCC-algebra X, we have*

$$(\forall x \in X)\ (\varphi(x) \geq 0). \tag{4}$$

Proof. For any $x \in X$, we have $\varphi(0) = \varphi(0 * x) \leq \varphi(0) + \varphi(x)$, and so $\varphi(x) \geq 0$. □

Table 1: $*$-operation

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Theorem 3.5. *Let S be a subalgebra of a BCC-algebra X. For any real numbers t_1 and t_2 with $0 \leq t_1 < t_2$, let φ_S be a real-valued function on X defined by*

$$\varphi_S(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. Then φ_S is a weak pseudo-valuation on X.

Proof. Straightforward. □

Given an element a of a BCC-algebra X, the set $A(a) := \{x \in X \mid x \leq a\}$ is called the *initial section* of X determined by a.

Corollary 3.6. *Let X be a BCC-algebra. For any $a \in X$, let φ be a real-valued function on X defined by*

$$\varphi_a(x) = \begin{cases} t_1 & \text{if } x \in A(a), \\ t_2 & \text{if } x \notin A(a) \end{cases}$$

for all $x \in X$ where t_1 and t_2 are real numbers with $t_2 > t_1 \geq 0$ and $A(a)$ is the initial section of X determined by a. Then φ_a is a weak pseudo-valuation on X.

Theorem 3.7. *In a BCC-algebra, every BCK-pseudo-valuation is a weak pseudo-valuation.*

Proof. Let φ be a *BCK*-pseudo valuation on a *BCC*-algebra X. Using (a2) and (C2), we have $((x * y) * x) * y = 0 * y = 0$ for all $x, y \in X$. Hence

$$\begin{aligned} 0 = \varphi(0) &= \varphi(((x * y) * x) * y) \\ &\geq \varphi((x * y) * x) - \varphi(y) \\ &\geq \varphi(x * y) - \varphi(x) - \varphi(y), \end{aligned}$$

and so $\varphi(x * y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in X$. Therefore φ is a weak pseudo-valuation on X. □

The following example shows that the converse of Theorem 3.7 is not true.

Example 3.8. Consider the BCC-algebra X which is given in Example 3.3. Let θ be a real-valued function on X defined by

$$\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 5 \end{pmatrix}.$$

It is easy to show that θ is a weak pseudo-valuation , but not a BCK-pseudo-valuation on X since

$$\theta(3) = 4 \nleq 3 = 1 + 2 = \theta(1) + \theta(2) = \theta(3 * 2) + \theta(2).$$

Definition 3.9. A real-valued function φ on a BCC-algebra X is called a *BCC-pseudo-valuation* on X if it satisfies (2) and

$$(\forall x, y, z \in X)\ (\varphi((x * y) * z) \geq \varphi(x * z) - \varphi(y)). \tag{5}$$

Example 3.10. Consider the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of natural numbers. Define a binary operation $*$ on $\mathbb{N}_0$ by

$$(\forall x, y \in \mathbb{N}_0)\left(x * y := \begin{cases} 0 & \text{if } x \leq y \\ x - y & \text{if } x > y \end{cases}\right).$$

Then $(\mathbb{N}_0; *, 0)$ is a BCK-algebra with the unique small atom 1, and so it is a BCC-algebra. Define

$$\varphi : \mathbb{N}_0 \to \mathbb{R},\ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 2x + 1 & \text{otherwise.} \end{cases}$$

It is routine to verify that φ is a BCC-pseudo-valuation on $\mathbb{N}_0$.

Putting $z = 0$ in (5) and using (C3), we get $\varphi(x * y) \geq \varphi(x) - \varphi(y)$ for all $x, y \in X$. Thus we know that every BCC-pseudo-valuation is a BCK-pseudo-valuation. We will state this as a theorem.

Theorem 3.11. *In a BCC-algebra, every BCC-pseudo-valuation is a BCK-pseudo-valuation.*

The converse of Theorem 3.11 is not true as seen in the following example.

Example 3.12. Consider the BCC-algebra X which is given in Example 3.3. Let φ be as in Example 3.3. Then φ is a BCK-pseudo-valuation, but not a BCC-pseudo-valuation on X since

$$\varphi((4 * 1) * 2) = \varphi(1) = 1 \not\geq 4 = \varphi(4 * 2) - \varphi(1).$$

Theorem 3.13. *In a BCK-algebra, every BCK-pseudo-valuation is a BCC-pseudo-valuation.*

Proof. Let φ be a BCK-pseudo-valuation on a BCK-algebra X and let $x, y, z \in X$. Then

$$\varphi(x * z) \leq \varphi((x * z) * y) + \varphi(y) = \varphi((x * y) * z) + \varphi(y)$$

and so φ is a BCC-pseudo-valuation on X. □

Lemma 3.14. *Let φ be a BCC-pseudo-valuation on a BCC-algebra X. If $x \leq y$ then $\varphi(x) \leq \varphi(y)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned}\varphi(x) &= \varphi(x * 0) \leq \varphi((x * y) * 0) + \varphi(y) \\ &= \varphi(x * y) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y).\end{aligned}$$

This completes the proof. □

Lemma 3.15. *Every BCC-pseudo-valuation on a BCC-algebra X is a weak pseudo-valuation on X.*

Proof. It is clear. □

Corollary 3.16. *Every BCC-pseudo-valuation on a BCC-algebra X satisfies the following assertions: for all $x, y, z \in X$,*

(a) $\varphi(x * y) \leq \varphi(x)$,
(b) $\varphi(x * (y * z)) \leq \varphi(x) + \varphi(y) + \varphi(z)$,
(c) $\varphi((x * y) * (z * y)) \leq \varphi(x * z)$,

(d) $x \le y \Rightarrow \varphi(x * z) \le \varphi(y * z),\ \varphi(z * y) \le \varphi(z * x)$.

The following example shows that the converse of Lemma 3.15 is not true.

Example 3.17. Consider the BCC-algebra X which is given in Example 3.3. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 3 \end{pmatrix}.$$

It is easy to check that φ is a weak pseudo-valuation, but not a BCK-pseudo-valuation since $\varphi(0) \neq 0$. Also it is not a BCC-pseudo-valuation since

$$\varphi((4 * 1) * 2) \not\geq \varphi(4 * 2) - \varphi(1).$$

Proposition 3.18. *Every BCC-pseudo-valuation on a BCC-algebra X satisfies the following implication:*

$$(\forall x, y, z, a \in X)\ ((x * y) * z \le a \Rightarrow \varphi(x * z) \le \varphi(y) + \varphi(a)). \tag{6}$$

Proof. Let $x, y, z, a \in X$ be such that $(x * y) * z \le a$. It follows from Lemma 3.14 that $\varphi((x * y) * z) \le \varphi(a)$ so from (5) that

$$\varphi(x * z) \le \varphi((x * y) * z) + \varphi(y) \le \varphi(a) + \varphi(y).$$

This completes the proof. □

We provide a condition for a real-valued function φ on a BCC-algebra X to be a BCC-pseudo-valuation on X.

Theorem 3.19. *Let φ be a real-valued function on a BCC-algebra X. If φ satisfies conditions* (2) *and* (6)*, then φ is a BCC-pseudo-valuation on X.*

Proof. Assume that φ satisfies conditions (2) and (6). We note that $(x * y) * z \le (x * y) * z$ for all $x, y, z \in X$, and so $\varphi(x * z) \le \varphi((x * y) * z) + \varphi(y)$. Therefore φ is a BCC-pseudo-valuation on X. □

Definition 3.20. A real-valued function φ on a BCC-algebra X is called a *strong BCC-pseudo-valuation* on X if it satisfies (2) and

$$(\forall x, y, z \in X)\ (\varphi((x * y) * z) \ge \varphi(x) - \varphi(y)). \tag{7}$$

Lemma 3.21. *Every strong BCC-pseudo-valuation φ on a BCC-algebra X is order preserving.*

Proof. Let $x, y \in X$ be such that $x \le y$. Then $x * y = 0$, and so

$$\varphi(x) \le \varphi((x * y) * 0) + \varphi(y) = \varphi(0 * 0) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y)$$

by (7), (2) and (a1). Hence φ is order preserving. □

Theorem 3.22. *Every strong BCC-pseudo-valuation φ on a BCC-algebra X is a BCC-pseudo-valuation on X.*

Proof. By (a2) and Lemma 3.21, we have $\varphi(x * z) \le \varphi(x)$ for all $x, z \in X$. It follows from (7) that

$$\varphi((x * y) * z) \ge \varphi(x) - \varphi(y) \ge \varphi(x * z) - \varphi(y). \tag{8}$$

Hence φ is a BCC-pseudo-valuation on X. □

The following example shows that the converse of Theorem 3.22 may not be true.

Table 2: $*$-operation

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Example 3.23. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra ([7]), which is not a BCK-algebra, with $*$-operation given by Table 2. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 1 & 7 \end{pmatrix}.$$

It is easy to check that φ is a BCC-pseudo-valuation on X, but not a strong BCC-pseudo-valuation on X, since $\varphi((1 * 0) * 1) = 0 \ngeq 1 = 1 - 0 = \varphi(1) - \varphi(0)$.

Definition 3.24 ([6]). A non-zero element a of a BCC-algebra X is called an *atom* of X if for any $x \in X$, $x \leq a$ implies $x = 0$ or $x = a$.

Lemma 3.25 ([6]). *Let a and b be atoms of a BCC-algebra X. If $a \neq b$, then $a * b = a$.*

We provide a condition for a BCC-pseudo-valuation to be a strong BCC-pseudo-valuation.

Theorem 3.26. *In a BCC-algebra containing only atoms, every BCC-pseudo-valuation is a strong BCC-pseudo-valuation.*

Proof. Let X be a BCC-algebra containing only atoms and let φ be a BCC-pseudo-valuation on X. Using Lemma 3.25 and (5), we have

$$\varphi(x) = \varphi(x * z) \leq \varphi((x * y) * z) + \varphi(y)$$

for all $x, y, z \in X$. Hence φ is a strong BCC-pseudo-valuation on X. □

Proposition 3.27. *For any BCK-pseudo-valuation φ on a BCC-algebra X, we have the following assertions:*

(a) *φ is order preserving,*

(b) $(\forall x, y \in X)(\varphi(x * y) + \varphi(y * x) \geq 0)$,

(c) $(\forall x, y, z \in X)(\varphi(x * y) \leq \varphi(x * z) + \varphi(z * y))$.

Proof. (a) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so $\varphi(x) \leq \varphi(x * y) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y)$.

(b) Let $x, y \in X$. Using (3), we have $\varphi(x * y) \geq \varphi(x) - \varphi(y)$ and $\varphi(y * x) \geq \varphi(y) - \varphi(x)$. It follows that $\varphi(x * y) + \varphi(y * x) \geq 0$.

(c) Let $x, y, z \in X$. Since φ is order preserving, it follows from (C1) and (3) that

$$\varphi(x * z) \geq \varphi((x * y) * (z * y)) \geq \varphi(x * y) - \varphi(z * y).$$

Hence (c) is valid. □

Corollary 3.28. *Every BCC-pseudo-valuation φ on a BCC-algebra X satisfies conditions* (a), (b) *and* (c) *in Proposition 3.27.*

Theorem 3.29. *If a real-valued function φ on a BCC-algebra X satisfies the condition* (2) *and*

$$(\forall x, y, z \in X)(\varphi(((x * y) * y) * z) \geq \varphi(x * y) - \varphi(z) \tag{9}$$

then φ is a BCK-pseudo-valuation on X.

Proof. Taking $y = 0$ in (9) and using (C3), we have

$$\varphi(x * z) = \varphi(((x * 0) * 0) * z) \geq \varphi(x * 0) - \varphi(z) = \varphi(x) - \varphi(z).$$

Hence φ is a BCK-pseudo-valuation on X. □

Corollary 3.30. *Let φ be a real-valued function on a BCK-algebra X. If φ satisfies conditions* (2) *and* (9), *then φ is a BCC-pseudo-valuation on X.*

By a *pseudo-metric space* we mean an ordered pair (M, d), where M is a non-empty set and $d : M \times M \to \mathbb{R}$ is a positive function satisfying the following properties: $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in M$. If in the pseudo-metric space (M, d) the implication $d(x, y) = 0 \Rightarrow x = y$ hold, then (M, d) is called a *metric space*. For a real-valued function φ on a BCC-algebra X, define a mapping $d_\varphi : X \times X \to \mathbb{R}$ by $d_\varphi(x, y) = \varphi(x * y) + \varphi(y * x)$ for all $(x, y) \in X \times X$.

Theorem 3.31. *If a real-valued function φ on a BCC-algebra X is a BCK-pseudo-valuation on X, then (X, d_φ) is a pseudo-metric space.*

We say d_φ is the *pseudo-metric* induced by a BCK-pseudo-valuation φ on a BCC-algebra X.

Proof. Obviously, $d_\varphi(x, y) \geq 0, d_\varphi(x, x) = 0$ and $d_\varphi(x, y) = d_\varphi(y, x)$ for all $x, y \in X$. Let $x, y, z \in X$. Using Proposition 3.27(c), we have

$$\begin{aligned} d_\varphi(x, y) + d_\varphi(y, z) &= [\varphi(x * y) + \varphi(y * x)] + [\varphi(y * z) + \varphi(z * y)] \\ &= [\varphi(x * y) + \varphi(y * z)] + [\varphi(z * y) + \varphi(y * x)] \\ &\geq \varphi(x * z) + \varphi(z * x) = d_\varphi(x, z). \end{aligned}$$

Therefore (X, d_φ) is a pseudo-metric space. □

The following example illustrates Theorem 3.31.

Example 3.32. Consider the BCC-pseudo-valuation φ on $\mathbb{N}_0$ which is described in Example 3.10. Using Theorem 3.11, we know that φ is a BCK-pseudo-valuation on $\mathbb{N}_0$. The pseudo-metric d_φ induced by φ is given as follows:

$$d_\varphi(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2y + 1 & \text{if } x = 0 \text{ and } y \neq 0, \\ 2x + 1 & \text{if } x \neq 0 \text{ and } y = 0, \\ 2(y * x) + 1 & \text{if } \begin{cases} x * y = 0 \\ y * x \neq 0 \end{cases} \text{ for } 0 \neq x \neq y \neq 0, \\ 2(x * y) + 1 & \text{if } \begin{cases} x * y \neq 0 \\ y * x = 0 \end{cases} \text{ for } 0 \neq x \neq y \neq 0, \\ 2(x * y) + 2(y * x) + 2 & \text{if } \begin{cases} x * y \neq 0 \\ y * x \neq 0 \end{cases} \text{ for } 0 \neq x \neq y \neq 0, \end{cases}$$

and $(\mathbb{N}_0, d_\varphi)$ is a pseudo-metric space.

Proposition 3.33. *Let φ be a BCK-pseudo-valuation on a BCC-algebra X. Then every pseudo-metric d_φ induced by φ satisfies the following inequalities:*

(a) $d_\varphi(x, y) \geq \max\{d_\varphi(x * a, y * a), d_\varphi(a * x, a * y)\}$,
(b) $d_\varphi(x * y, a * b) \leq d_\varphi(x * y, a * y) + d_\varphi(a * y, a * b)$

for all $x, y, a, b \in X$.

Proof. (a) Let $x, y, a \in X$. Since

$$((y * a) * (x * a)) * (y * x) = 0 \text{ and } ((x * a) * (y * a)) * (x * y) = 0,$$

it follows from Proposition 3.27(a) that $\varphi(y * x) \geq \varphi((y * a) * (x * a))$ and $\varphi(x * y) \geq \varphi((x * a) * (y * a))$ so that

$$\begin{aligned} d_\varphi(x, y) &= \varphi(x * y) + \varphi(y * x) \\ &\geq \varphi((x * a) * (y * a)) + \varphi((y * a) * (x * a)) \\ &= d_\varphi(x * a, y * a). \end{aligned}$$

Similarly, we have $d_\varphi(x, y) \geq d_\varphi(a * x, a * y)$. Hence (a) is valid.

(b) Using Proposition 3.27(c), we have

$$\varphi((x * y) * (a * b)) \leq \varphi((x * y) * (a * y)) + \varphi((a * y) * (a * b)),$$

$$\varphi((a * b) * (x * y)) \leq \varphi((a * b) * (a * y)) + \varphi((a * y) * (x * y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_\varphi(x * y, a * b) &= \varphi((x * y) * (a * b)) + \varphi((a * b) * (x * y)) \\ &\leq [\varphi((x * y) * (a * y)) + \varphi((a * y) * (a * b))] \\ &\quad + [\varphi((a * b) * (a * y)) + \varphi((a * y) * (x * y))] \\ &= [\varphi((x * y) * (a * y)) + \varphi((a * y) * (x * y))] \\ &\quad + [\varphi((a * b) * (a * y)) + \varphi((a * y) * (a * b))] \\ &= d_\varphi(x * y, a * y) + d_\varphi(a * y, a * b) \end{aligned}$$

for all $x, y, a, b \in X$. □

Theorem 3.34. *For a real-valued function φ on a BCC-algebra X, if d_φ is a pseudo-metric on X, then $(X \times X, d^*_\varphi)$ is a pseudo-metric space, where*

$$d^*_\varphi((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\} \tag{10}$$

for all $(x, y), (a, b) \in X \times X$.

Proof. Suppose d_φ is a pseudo-metric on X. For any $(x, y), (a, b) \in X \times X$, we have

$$d^*_\varphi((x, y), (x, y)) = \max\{d_\varphi(x, x), d_\varphi(y, y)\} = 0$$

and

$$\begin{aligned} d^*_\varphi((x, y), (a, b)) &= \max\{d_\varphi(x, a), d_\varphi(y, b)\} \\ &= \max\{d_\varphi(a, x), d_\varphi(b, y)\} \\ &= d^*_\varphi((a, b), (x, y)). \end{aligned}$$

Now let $(x, y), (a, b), (u, v) \in X \times X$. Then

$$\begin{aligned}
&d^*_\varphi((x, y), (u, v)) + d^*_\varphi((u, v), (a, b)) \\
&= \max\{d_\varphi(x, u), d_\varphi(y, v)\} + \max\{d_\varphi(u, a), d_\varphi(v, b)\} \\
&\geq \max\{d_\varphi(x, u) + d_\varphi(u, a), d_\varphi(y, v) + d_\varphi(v, b)\} \\
&\geq \max\{d_\varphi(x, a), d_\varphi(y, b)\} \\
&= d^*_\varphi((x, y), (a, b)).
\end{aligned}$$

Therefore $(X \times X, d^*_\varphi)$ is a pseudo-metric space. □

Corollary 3.35. *If $\varphi : X \to \mathbb{R}$ is a BCK-pseudo-valuation on a BCC-algebra X, then $(X \times X, d^*_\varphi)$ is a pseudo-metric space.*

A BCK/BCC-pseudo-valuation φ on a BCC-algebra X satisfying the following condition:

$$(\forall x \in X)\ (x \neq 0 \ \Rightarrow\ \varphi(x) \neq 0) \tag{11}$$

is called a *BCK/BCC-valuation* on X.

Theorem 3.36. *If $\varphi : X \to \mathbb{R}$ is a BCK-valuation on a BCC-algebra X, then (X, d_φ) is a metric space.*

Proof. Suppose φ is a BCK-valuation on a BCC-algebra X. Then (X, d_φ) is a pseudo-metric space by Theorem 3.31. Let $x, y \in X$ be such that $d_\varphi(x, y) = 0$. Then $0 = d_\varphi(x, y) = \varphi(x * y) + \varphi(y * x)$, and so $\varphi(x * y) = 0$ and $\varphi(y * x) = 0$. It follows from (11) that $x * y = 0$ and $y * x = 0$ so from (C4) that $x = y$. Therefore (X, d_φ) is a metric space. □

Theorem 3.37. *If $\varphi : X \to \mathbb{R}$ is a BCK-valuation on a BCC-algebra X, then $(X \times X, d^*_\varphi)$ is a metric space.*

Proof. Note from Corollary 3.35 that $(X \times X, d^*_\varphi)$ is a pseudo-metric space. Let $(x, y), (a, b) \in X \times X$ be such that $d^*_\varphi((x, y), (a, b)) = 0$. Then

$$0 = d^*_\varphi((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\},$$

and so $d_\varphi(x, a) = 0 = d_\varphi(y, b)$ since $d_\varphi(x, y) \geq 0$ for all $(x, y) \in X \times X$. Hence

$$0 = d_\varphi(x, a) = \varphi(x * a) + \varphi(a * x)$$

and

$$0 = d_\varphi(y, b) = \varphi(y * b) + \varphi(b * y).$$

It follows that $\varphi(x * a) = 0 = \varphi(a * x)$ and $\varphi(y * b) = 0 = \varphi(b * y)$ so that $x * a = 0 = a * x$ and $y * b = 0 = b * y$. Using (C4), we have $a = x$ and $b = y$, and so $(x, y) = (a, b)$. Therefore $(X \times X, d^*_\varphi)$ is a metric space. □

Theorem 3.38. *If $\varphi : X \to \mathbb{R}$ is a BCK-valuation on a BCC-algebra X, then the operation $*$ in the BCC-algebra X is uniformly continuous.*

Proof. For any $\varepsilon > 0$, if $d^*_\varphi((x, y), (a, b)) < \frac{\varepsilon}{2}$, then $d_\varphi(x, a) < \frac{\varepsilon}{2}$ and $d_\varphi(y, b) < \frac{\varepsilon}{2}$. Using Proposition 3.33, we have

$$\begin{aligned}
d_\varphi(x * y, y * a) &\leq d_\varphi((x, y), (a * y) + d_\varphi(a * y, a * b) \\
&\leq d_\varphi(x, a) + d_\varphi(y, b) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore the operation $* : X \times X \to X$ is uniformly continuous. □

Table 3: $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	b	b	0

The following example illustrates Theorem 3.38.

Example 3.39. Let $X = \{0, a, b, c\}$ be a set with the $*$-operation given by Table 3. Then $(X, *, 0)$ is a proper BCC-algebra. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 4 & 5 \end{pmatrix}.$$

Then φ is a BCK-valuation on X and (X, d_φ) is a metric space where

$$d_\varphi = \begin{pmatrix} (0,0) & (0,a) & (0,b) & (0,c) & (a,a) & (a,b) & (a,c) & (b,b) & (b,c) & (c,c) \\ 0 & 3 & 4 & 5 & 0 & 3 & 4 & 0 & 4 & 0 \end{pmatrix}.$$

Also, $(X \times X, d^*_\varphi)$ is a metric space where d^*_φ is obtained by (10), for example,

$$d^*_\varphi((0,b),(a,c)) = \max\{d_\varphi(0,a), d_\varphi(b,c)\} = \max\{3,4\} = 4,$$
$$d^*_\varphi((a,b),(c,a)) = \max\{d_\varphi(a,c), d_\varphi(b,a)\} = \max\{4,3\} = 4,$$
$$d^*_\varphi((c,a),(0,0)) = \max\{d_\varphi(c,0), d_\varphi(a,0)\} = \max\{5,3\} = 5,$$
$$d^*_\varphi((a,c),(b,0)) = \max\{d_\varphi(a,b), d_\varphi(c,0)\} = \max\{3,5\} = 5,$$
$$d^*_\varphi((a,c),(b,c)) = \max\{d_\varphi(a,b), d_\varphi(c,c)\} = \max\{3,0\} = 3,$$

and so on. Now, it is routine to verify that the operation $*$ in the BCC-algebra X

$$* : X \times X \to X,\ (x, y) \mapsto x * y$$

is uniformly continuous.

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