BCC-algebras with pseudo-valuations

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Abstract. The notion of pseudo-valuations (valuations) on a BCC-algebra is introduced by using the Bușneag’s model ([1–3]), and a pseudo-metric is induced by a pseudo-valuation on BCC-algebras. Conditions for a real-valued function to be an BCK-pseudo-valuation are provided. The fact that the binary operation in BCC-algebras is uniformly continuous is provided based on the notion of (pseudo) valuation.

1. Introduction

In 1966, Y. Imai and K. Iséki (cf. [8]) defined a class of algebras of type (2, 0) called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (cf. [8]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [12]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori (cf. [10]) introduced a notion of BCC-algebras, and W. A. Dudek (cf. [4, 5]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [7], W. A. Dudek and X. H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. Bușneag [2] defined a pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Also, Bușneag [3] provided several theorems on extensions of pseudo-valuations. Bușneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras ([3])).

In this paper, using the Bușneag’s model, we introduce the notion of (BCK, BCC, strong BCC)-pseudo-valuations (valuations) on BCC-algebras, and we induce a pseudo-metric by using a BCK-pseudo-valuation on BCC-algebras. We provide conditions for a real-valued function on a BCC-algebra X to be a BCK-pseudo-pseudo-valuation on X. Based on the notion of (pseudo) valuation, we show that the binary operation * in BCC-algebras is uniformly continuous.
2. Preliminaries

Recall that a BCC-algebra is an algebra \((X, *, 0)\) of type (2,0) satisfying the following axioms:

\begin{align*}
(C1) \quad & ((x * y) * (z * y)) * (x * z) = 0, \\
(C2) \quad & 0 * x = 0, \\
(C3) \quad & x * 0 = x, \\
(C4) \quad & x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y
\end{align*}

for every \(x, y, z \in X\). For any BCC-algebra \(X\), the relation \(\leq\) defined by \(x \leq y\) if and only if \(x * y = 0\) is a partial order on \(X\). In a BCC-algebra \(X\), the following holds:

\begin{align*}
(a1) \quad & (\forall x \in X) (x * x = 0), \\
(a2) \quad & (\forall x, y \in X) (x * y \leq x), \\
(a3) \quad & (\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x).
\end{align*}

A subset \(I\) of a BCC-algebra \(X\) is called a BCK-ideal if it satisfies:

\begin{align*}
(i) \quad & 0 \in I, \\
(ii) \quad & (\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I).
\end{align*}

A subset \(I\) of a BCC-algebra \(X\) is called a BCC-ideal if it satisfies:

\begin{align*}
(i) \quad & 0 \in I, \\
(ii) \quad & (\forall x, z \in X) (\forall y \in I) ((x * y) * z \in I \Rightarrow x * z \in I).
\end{align*}

3. Pseudo-valuations on BCC-algebras

**Definition 3.1.** A real-valued function \(\varphi\) on a BCC-algebra \(X\) is called a weak pseudo-valuation on \(X\) if it satisfies the following condition:

\[(\forall x, y \in X) (\varphi(x * y) \leq \varphi(x) + \varphi(y)).\]  \tag{1}

**Definition 3.2.** A real-valued function \(\varphi\) on a BCC-algebra \(X\) is called a BCK-pseudo-valuation on \(X\) if it satisfies the following condition:

\[
\begin{align*}
\varphi(0) = 0, \\
(\forall x, y \in X) (\varphi(x * y) \geq \varphi(x) - \varphi(y)). \tag{2}
\end{align*}
\]

**Example 3.3.** Let \(X := \{0, 1, 2, 3, 4\}\) be a BCC-algebra ([7]), which is not a BCK-algebra, with \(*\)-operation given by Table 1. Let \(\varphi\) be a real-valued function on \(X\) defined by

\[
\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 4 & 5 \end{pmatrix}.
\]

It is easy to check that \(\varphi\) is both a weak pseudo-valuation and a BCK-pseudo-valuation on \(X\).

**Proposition 3.4.** For a weak pseudo-valuation \(\varphi\) on a BCC-algebra \(X\), we have

\[(\forall x \in X) (\varphi(x) \geq 0).\]  \tag{4}

**Proof.** For any \(x \in X\), we have \(\varphi(0) = \varphi(0 * x) \leq \varphi(0) + \varphi(x)\), and so \(\varphi(x) \geq 0\). \(\Box\)
Theorem 3.5. Let $S$ be a subalgebra of a BCC-algebra $X$. For any real numbers $t_1$ and $t_2$ with $0 \leq t_1 < t_2$, let $\varphi_S$ be a real-valued function on $X$ defined by

$$\varphi_S(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. Then $\varphi_S$ is a weak pseudo-valuation on $X$.

Proof. Straightforward.

Given an element $a$ of a BCC-algebra $X$, the set $A(a) := \{x \in X \mid x \leq a\}$ is called the initial section of $X$ determined by $a$.

Corollary 3.6. Let $X$ be a BCC-algebra. For any $a \in X$, let $\varphi_a$ be a real-valued function on $X$ defined by

$$\varphi_a(x) = \begin{cases} t_1 & \text{if } x \in A(a), \\ t_2 & \text{if } x \notin A(a) \end{cases}$$

for all $x \in X$ where $t_1$ and $t_2$ are real numbers with $t_2 > t_1 \geq 0$ and $A(a)$ is the initial section of $X$ determined by $a$. Then $\varphi_a$ is a weak pseudo-valuation on $X$.

Theorem 3.7. In a BCC-algebra, every BCK-pseudo-valuation is a weak pseudo-valuation.

Proof. Let $\varphi$ be a BCK-pseudo valuation on a BCC-algebra $X$. Using (a2) and (C2), we have $(x * y) * x = 0 * y = 0$ for all $x, y \in X$. Hence

$$0 = \varphi(0) = \varphi(((x * y) * x) * y) \geq \varphi((x * y) * x) - \varphi(y) \geq \varphi(x * y) - \varphi(x) - \varphi(y),$$

and so $\varphi(x * y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in X$. Therefore $\varphi$ is a weak pseudo-valuation on $X$. \qed

The following example shows that the converse of Theorem 3.7 is not true.

Example 3.8. Consider the BCC-algebra $X$ which is given in Example 3.3. Let $\theta$ be a real-valued function on $X$ defined by

$$\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 5 \end{pmatrix}.$$  

It is easy to show that $\theta$ is a weak pseudo-valuation, but not a BCK-pseudo-valuation on $X$ since

$$\theta(3) = 4 \not\leq 3 = 1 + 2 = \theta(1) + \theta(2) = \theta(3 * 2) + \theta(2).$$

<table>
<thead>
<tr>
<th>Table 1: $\ast$-operation</th>
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Definition 3.9. A real-valued function $\varphi$ on a BCC-algebra $X$ is called a BCC-pseudo-valuation on $X$ if it satisfies (2) and

$$(\forall x, y, z \in X) (\varphi((x \ast y) \ast z) \geq \varphi(x \ast z) - \varphi(y)).$$

Example 3.10. Consider the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of natural numbers. Define a binary operation $\ast$ on $\mathbb{N}_0$ by

$$(\forall x, y \in \mathbb{N}_0) (x \ast y := \begin{cases} 0 & \text{if } x \leq y \\ x - y & \text{if } x > y \end{cases}).$$

Then $(\mathbb{N}_0; \ast, 0)$ is a BCK-algebra with the unique small atom 1, and so it is a BCC-algebra. Define

$$\varphi : \mathbb{N}_0 \rightarrow \mathbb{R}, \ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 2x + 1 & \text{otherwise}. \end{cases}$$

It is routine to verify that $\varphi$ is a BCC-pseudo-valuation on $\mathbb{N}_0$.

Putting $z = 0$ in (5) and using (C3), we get $\varphi(x \ast y) \geq \varphi(x) - \varphi(y)$ for all $x, y \in X$. Thus we know that every BCC-pseudo-valuation is a BCK-pseudo-valuation. We will state this as a theorem.

Theorem 3.11. In a BCC-algebra, every BCC-pseudo-valuation is a BCK-pseudo-valuation.

The converse of Theorem 3.11 is not true as seen in the following example.

Example 3.12. Consider the BCC-algebra $X$ which is given in Example 3.3. Let $\varphi$ be as in Example 3.3. Then $\varphi$ is a BCK-pseudo-valuation, but not a BCC-pseudo-valuation on $X$ since

$$\varphi((4 \ast 1) \ast 2) = \varphi(1) = 1 \not\geq 4 = \varphi(4 \ast 2) - \varphi(1).$$

Theorem 3.13. In a BCK-algebra, every BCK-pseudo-valuation is a BCC-pseudo-valuation.

Proof. Let $\varphi$ be a BCK-pseudo-valuation on a BCK-algebra $X$ and let $x, y, z \in X$. Then

$$\varphi(x \ast y) \leq \varphi((x \ast z) \ast y) + \varphi(y) = \varphi((x \ast y) \ast z) + \varphi(y)$$

and so $\varphi$ is a BCC-pseudo-valuation on $X$. $\square$

Lemma 3.14. Let $\varphi$ be a BCC-pseudo-valuation on a BCC-algebra $X$. If $x \leq y$ then $\varphi(x) \leq \varphi(y)$ for all $x, y \in X$.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \ast y = 0$, and so

$$\varphi(x) = \varphi(x \ast 0) \leq \varphi((x \ast y) \ast 0) + \varphi(y) = \varphi(x \ast y) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y).$$

This completes the proof. $\square$

Lemma 3.15. Every BCC-pseudo-valuation on a BCC-algebra $X$ is a weak pseudo-valuation on $X$.

Proof. It is clear. $\square$

Corollary 3.16. Every BCC-pseudo-valuation on a BCC-algebra $X$ satisfies the following assertions: for all $x, y, z \in X$,

(a) $\varphi(x \ast y) \leq \varphi(x)$,
(b) $\varphi(x \ast (y \ast z)) \leq \varphi(x) + \varphi(y) + \varphi(z)$,
(c) $\varphi((x \ast y) \ast (z \ast y)) \leq \varphi(x \ast z)$,
(d) \( x \leq y \Rightarrow \varphi(x * z) \leq \varphi(y * z), \\varphi(z * y) \leq \varphi(z * x). \)

The following example shows that the converse of Lemma 3.15 is not true.

**Example 3.17.** Consider the BCC-algebra \( X \) which is given in Example 3.3. Let \( \varphi \) be a real-valued function on \( X \) defined by

\[
\varphi = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 3
\end{pmatrix}.
\]

It is easy to check that \( \varphi \) is a weak pseudo-valuation, but not a BCK-pseudo-valuation since \( \varphi(0) \neq 0 \). Also it is not a BCC-pseudo-valuation since

\[
\varphi((4 * 1) * 2) \neq \varphi(4 * 2) - \varphi(1).
\]

**Proposition 3.18.** *Every BCC-pseudo-valuation on a BCC-algebra \( X \) satisfies the following implication:*

\[
(\forall x, y, z, a \in X) ((x * y) * z \leq a \Rightarrow \varphi(x * z) \leq \varphi(y) + \varphi(a)).
\]

**Proof.** Let \( x, y, z, a \in X \) be such that \((x * y) * z \leq a\). It follows from Lemma 3.14 that \( \varphi((x * y) * z) \leq \varphi(a) \) so from (5) that

\[
\varphi(x * z) \leq \varphi((x * y) * z) + \varphi(y) \leq \varphi(a) + \varphi(y).
\]

This completes the proof. □

We provide a condition for a real-valued function \( \varphi \) on a BCC-algebra \( X \) to be a BCC-pseudo-valuation on \( X \).

**Theorem 3.19.** Let \( \varphi \) be a real-valued function on a BCC-algebra \( X \). If \( \varphi \) satisfies conditions (2) and (6), then \( \varphi \) is a BCC-pseudo-valuation on \( X \).

**Proof.** Assume that \( \varphi \) satisfies conditions (2) and (6). We note that \( (x * y) * z \leq (x * y) * z \) for all \( x, y, z \in X \), and so \( \varphi(x * z) \leq \varphi((x * y) * z) + \varphi(y) \). Therefore \( \varphi \) is a BCC-pseudo-valuation on \( X \). □

**Definition 3.20.** A real-valued function \( \varphi \) on a BCC-algebra \( X \) is called a *strong BCC-pseudo-valuation* on \( X \) if it satisfies (2) and

\[
(\forall x, y, z \in X) (\varphi((x * y) * z) \geq \varphi(x) - \varphi(y)).
\]

**Lemma 3.21.** *Every strong BCC-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \) is order preserving.*

**Proof.** Let \( x, y \in X \) be such that \( x \leq y \). Then \( x * y = 0 \), and so

\[
\varphi(x) \leq \varphi((x * y) * 0) + \varphi(y) = \varphi(0 * 0) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y)
\]

by (7), (2) and (a1). Hence \( \varphi \) is order preserving. □

**Theorem 3.22.** *Every strong BCC-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \) is a BCC-pseudo-valuation on \( X \).*

**Proof.** By (a2) and Lemma 3.21, we have \( \varphi(x * z) \leq \varphi(x) \) for all \( x, z \in X \). It follows from (7) that

\[
\varphi((x * y) * z) \geq \varphi(x) - \varphi(y) \geq \varphi(x * z) - \varphi(y).
\]

Hence \( \varphi \) is a BCC-pseudo-valuation on \( X \). □

The following example shows that the converse of Theorem 3.22 may not be true.
Table 2: $\ast$-operation

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<thead>
<tr>
<th></th>
<th>0</th>
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<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

**Example 3.23.** Let $X := \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra ([7]), which is not a BCK-algebra, with $\ast$-operation given by Table 2. Let $\varphi$ be a real-valued function on $X$ defined by

$$
\varphi = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 1 & 1 & 1 & 7
\end{pmatrix}.
$$

It is easy to check that $\varphi$ is a BCC-pseudo-valuation on $X$, but not a strong BCC-pseudo-valuation on $X$, since $\varphi((1 \ast 0) \ast 1) = 0 \not\geq 1 - 0 = \varphi(1) - \varphi(0)$.

**Definition 3.24 ([6]).** A non-zero element $a$ of a BCC-algebra is called an atom of $X$ if for any $x \in X$, $x \leq a$ implies $x = 0$ or $x = a$.

**Lemma 3.25 ([6]).** Let $a$ and $b$ be atoms of a BCC-algebra $X$. If $a \neq b$, then $a \ast b = a$.

We provide a condition for a BCC-pseudo-valuation to be a strong BCC-pseudo-valuation.

**Theorem 3.26.** In a BCC-algebra containing only atoms, every BCC-pseudo-valuation is a strong BCC-pseudo-valuation.

**Proof.** Let $X$ be a BCC-algebra containing only atoms and let $\varphi$ be a BCC-pseudo-valuation on $X$. Using Lemma 3.25 and (5), we have

$$
\varphi(x) = \varphi(x \ast z) \leq \varphi((x \ast y) \ast z) + \varphi(y)
$$

for all $x, y, z \in X$. Hence $\varphi$ is a strong BCC-pseudo-valuation on $X$. □

**Proposition 3.27.** For any BCK-pseudo-valuation $\varphi$ on a BCC-algebra $X$, we have the following assertions:

(a) $\varphi$ is order preserving,

(b) $(\forall x, y \in X)(\varphi(x \ast y) + \varphi(y \ast x) \geq 0),$

(c) $(\forall x, y, z \in X)(\varphi(x \ast y) \leq \varphi(x \ast z) + \varphi(z \ast y)).$

**Proof.** (a) Let $x, y \in X$ be such that $x \leq y$. Then $x \ast y = 0$, and so $\varphi(x) \leq \varphi(x \ast y) + \varphi(y) = \varphi(0) + \varphi(y) = \varphi(y)$.

(b) Let $x, y \in X$. Using (3), we have $\varphi(x \ast y) \geq \varphi(x) - \varphi(y)$ and $\varphi(y \ast x) \geq \varphi(y) - \varphi(x)$. It follows that $\varphi(x \ast y) + \varphi(y \ast x) \geq 0$.

(c) Let $x, y, z \in X$. Since $\varphi$ is order preserving, it follows from (C1) and (3) that

$$
\varphi(x \ast z) \geq \varphi((x \ast y) \ast (z \ast y)) \geq \varphi(x \ast y) - \varphi(z \ast y).
$$

Hence (c) is valid. □
**Corollary 3.28.** Every BCC-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \) satisfies conditions (a), (b) and (c) in Proposition 3.27.

**Theorem 3.29.** If a real-valued function \( \varphi \) on a BCC-algebra \( X \) satisfies the condition (2) and

\[
(Vx, y, z \in X)(\varphi((x \ast y) \ast z) \geq \varphi(x \ast y) - \varphi(z)) \tag{9}
\]

then \( \varphi \) is a BCK-pseudo-valuation on \( X \).

**Proof.** Taking \( y = 0 \) in (9) and using (C3), we have

\[
\varphi(x \ast z) = \varphi((x \ast 0) \ast z) \geq \varphi(x \ast 0) - \varphi(z) = \varphi(x) - \varphi(z).
\]

Hence \( \varphi \) is a BCK-pseudo-valuation on \( X \). \( \square \)

**Corollary 3.30.** Let \( \varphi \) be a real-valued function on a BCK-algebra \( X \). If \( \varphi \) satisfies conditions (2) and (9), then \( \varphi \) is a BCC-pseudo-valuation on \( X \).

By a pseudo-metric space we mean an ordered pair \((M,d)\), where \( M \) is a non-empty set and \( d : M \times M \to \mathbb{R} \) is a positive function satisfying the following properties: \( d(x,x) = 0, d(x,y) = d(y,x) \) and \( d(x,z) \leq d(x,y) + d(y,z) \) for every \( x, y, z \in M \). If in the pseudo-metric space \((M,d)\) the implication \( d(x,y) = 0 \Rightarrow x = y \) hold, then \((M,d)\) is called a metric space. For a real-valued function \( \varphi \) on a BCC-algebra \( X \), define a mapping \( d_\varphi : X \times X \to \mathbb{R} \) by \( d_\varphi(x,y) = \varphi(x \ast y) + \varphi(y \ast x) \) for all \((x,y) \in X \times X\).

**Theorem 3.31.** If a real-valued function \( \varphi \) on a BCC-algebra \( X \) is a BCK-pseudo-valuation on \( X \), then \((X,d_\varphi)\) is a pseudo-metric space.

We say \( d_\varphi \) is the pseudo-metric induced by a BCK-pseudo-valuation \( \varphi \) on a BCC-algebra \( X \).

**Proof.** Obviously, \( d_\varphi(x,y) \geq 0, d_\varphi(x,x) = 0 \) and \( d_\varphi(x,y) = d_\varphi(y,x) \) for all \( x, y \in X \). Let \( x, y, z \in X \). Using Proposition 3.27(c), we have

\[
d_\varphi(x,y) + d_\varphi(y,z) = [\varphi(x \ast y) + \varphi(y \ast x)] + [\varphi(y \ast z) + \varphi(z \ast y)]
\]

\[
= [\varphi(x \ast y) + \varphi(y \ast z)] + [\varphi(z \ast y) + \varphi(y \ast x)]
\]

\[
\geq \varphi(x \ast z) + \varphi(z \ast x) = d_\varphi(x,z).
\]

Therefore \((X,d_\varphi)\) is a pseudo-metric space. \( \square \)

The following example illustrates Theorem 3.31.

**Example 3.32.** Consider the BCC-pseudo-valuation \( \varphi \) on \( \mathbb{N}_0 \) which is described in Example 3.10. Using Theorem 3.11, we know that \( \varphi \) is a BCK-pseudo-valuation on \( \mathbb{N}_0 \). The pseudo-metric \( d_\varphi \) induced by \( \varphi \) is given as follows:

\[
d_\varphi(x,y) = \begin{cases} 
0 & \text{if } x = y, \\
2y + 1 & \text{if } x = 0 \text{ and } y \neq 0, \\
2x + 1 & \text{if } x \neq 0 \text{ and } y = 0, \\
2(y \ast x) + 1 & \text{if } \begin{cases} x \ast y = 0 \\
y \ast x \neq 0 \end{cases} \text{ for } 0 \neq x \neq y \neq 0, \\
2(x \ast y) + 1 & \text{if } \begin{cases} x \ast y \neq 0 \\
y \ast x = 0 \end{cases} \text{ for } 0 \neq x \neq y \neq 0, \\
2(x \ast y) + 2(y \ast x) + 2 & \text{if } \begin{cases} x \ast y \neq 0 \\
y \ast x \neq 0 \end{cases} \text{ for } 0 \neq x \neq y \neq 0,
\end{cases}
\]

and \((\mathbb{N}_0,d_\varphi)\) is a pseudo-metric space.
Proposition 3.33. Let \( \varphi \) be a BCK-pseudo-valuation on a BCC-algebra \( X \). Then every pse udo-metric \( d_\varphi \) induced by \( \varphi \) satisfies the following inequalities:

(a) \( d_\varphi(x, y) \geq \max\{d_\varphi(x \ast a, y \ast a), d_\varphi(a \ast x, a \ast y)\} \)
(b) \( d_\varphi(x \ast y, a \ast b) \leq d_\varphi(x \ast y, a \ast y) + d_\varphi(a \ast y, a \ast b) \)

for all \( x, y, a, b \in X \).

Proof. (a) Let \( x, y, a \in X \). Since

\[
((y \ast a) \ast (x \ast a)) \ast (y \ast x) = 0 \text{ and } ((x \ast a) \ast (y \ast a)) \ast (x \ast y) = 0,
\]

it follows from Proposition 3.27(a) that \( \varphi(y \ast x) \geq \varphi((y \ast a) \ast (x \ast a)) \) and \( \varphi(x \ast y) \geq \varphi((x \ast a) \ast (y \ast a)) \) so that

\[
d_\varphi(x, y) = \varphi(x \ast y) + \varphi(y \ast x) \\
\geq \varphi((x \ast a) \ast (y \ast a)) + \varphi((y \ast a) \ast (x \ast a)) \\
= d_\varphi(x \ast a, y \ast a).
\]

Similarly, we have \( d_\varphi(x, y) \geq d_\varphi(a \ast x, a \ast y) \). Hence (a) is valid.

(b) Using Proposition 3.27(c), we have

\[
\varphi((x \ast y) \ast (a \ast b)) \leq \varphi((x \ast y) \ast (a \ast y)) + \varphi((a \ast y) \ast (a \ast b)),
\]

\[
\varphi((a \ast b) \ast (x \ast y)) \leq \varphi((a \ast b) \ast (a \ast y)) + \varphi((a \ast y) \ast (x \ast y))
\]

for all \( x, y, a, b \in X \). Hence

\[
d_\varphi(x \ast y, a \ast b) = \varphi((x \ast y) \ast (a \ast b)) + \varphi((a \ast b) \ast (x \ast y)) \\
\leq [\varphi((x \ast y) \ast (a \ast y)) + \varphi((a \ast y) \ast (a \ast b))] \\
+ [\varphi((a \ast b) \ast (a \ast y)) + \varphi((a \ast y) \ast (x \ast y))] \\
= [\varphi((x \ast y) \ast (a \ast y)) + \varphi((a \ast y) \ast (x \ast y))] \\
+ [\varphi((a \ast b) \ast (a \ast y)) + \varphi((a \ast y) \ast (a \ast b))] \\
= d_\varphi(x \ast y, a \ast y) + d_\varphi(a \ast y, a \ast b)
\]

for all \( x, y, a, b \in X \). \( \square \)

Theorem 3.34. For a real-valued function \( \varphi \) on a BCC-algebra \( X \), if \( d_\varphi \) is a pseudo-metric on \( X \), then \( (X \times X, d_\varphi) \) is a pseudo-metric space, where

\[
d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\} \tag{10}
\]

for all \( (x, y), (a, b) \in X \times X \).

Proof. Suppose \( d_\varphi \) is a pseudo-metric on \( X \). For any \( (x, y), (a, b) \in X \times X \), we have

\[
d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\} = 0
\]

and

\[
d_\varphi^*((x, y), (a, b)) = \max\{d_\varphi(x, a), d_\varphi(y, b)\} \\
= \max\{d_\varphi(a, x), d_\varphi(b, y)\} \\
= d_\varphi^*((a, b), (x, y)).
\]

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Now let \((x, y), (a, b), (u, v) \in X \times X\). Then

\[
d^*_p((x, y), (u, v)) + d^*_p((u, v), (a, b)) = \max\{d_p(x, u), d_p(y, v)\} + \max\{d_p(u, a), d_p(v, b)\} \\
\geq \max\{d_p(x, u) + d_p(u, a), d_p(y, v) + d_p(v, b)\} \\
\geq \max\{d_p(x, a), d_p(y, b)\} \\
= d^*_p((x, y), (a, b)).
\]

Therefore \((X \times X, d^*_p)\) is a pseudo-metric space. 

**Corollary 3.35.** If \(\varphi : X \to \mathbb{R}\) is a BCK-pseudo-valuation on a BCC-algebra \(X\), then \((X \times X, d^*_p)\) is a pseudo-metric space.

A BCK/BCC-pseudo-valuation \(\varphi\) on a BCC-algebra \(X\) satisfying the following condition:

\[
(\forall x \in X) \ (x \neq 0 \implies \varphi(x) \neq 0) \tag{11}
\]

is called a BCK/BCC-valuation on \(X\).

**Theorem 3.36.** If \(\varphi : X \to \mathbb{R}\) is a BCK-valuation on a BCC-algebra \(X\), then \((X, d_\varphi)\) is a metric space.

**Proof.** Suppose \(\varphi\) is a BCK-valuation on a BCC-algebra \(X\). Then \((X, d_\varphi)\) is a pseudo-metric space by Theorem 3.31. Let \(x, y \in X\) be such that \(d_\varphi(x, y) = 0\). Then 0 = \(d_\varphi(x, y) = \varphi(x \ast y) + \varphi(y \ast x)\), and so \(\varphi(x \ast y) = 0\) and \(\varphi(y \ast x) = 0\). It follows from (11) that \(x \ast y = 0\) and \(y \ast x = 0\) so from (C4) that \(x = y\). Therefore \((X, d_\varphi)\) is a metric space. 

**Theorem 3.37.** If \(\varphi : X \to \mathbb{R}\) is a BCK-valuation on a BCC-algebra \(X\), then \((X \times X, d^*_p)\) is a metric space.

**Proof.** Note from Corollary 3.35 that \((X \times X, d^*_p)\) is a pseudo-metric space. Let \((x, y), (a, b) \in X \times X\) be such that \(d^*_p((x, y), (a, b)) = 0\). Then

\[
0 = d^*_p((x, y), (a, b)) = \max\{d_p(x, a), d_p(y, b)\},
\]

and so \(d_p(x, a) = 0 = d_p(y, b)\) since \(d_p(x, y) = 0\) for all \((x, y) \in X \times X\). Hence

\[
0 = d_p(x, a) = \varphi(x \ast a) + \varphi(a \ast x)
\]

and

\[
0 = d_p(y, b) = \varphi(y \ast b) + \varphi(b \ast y).
\]

It follows that \(\varphi(x \ast a) = 0 = \varphi(a \ast x)\) and \(\varphi(y \ast b) = 0 = \varphi(b \ast y)\) so that \(x \ast a = 0 = a \ast x\) and \(y \ast b = 0 = b \ast y\). Using (C4), we have \(a = x\) and \(b = y\), and so \((x, y) = (a, b)\). Therefore \((X \times X, d^*_p)\) is a metric space. 

**Theorem 3.38.** If \(\varphi : X \to \mathbb{R}\) is a BCK-valuation on a BCC-algebra \(X\), then the operation \(*\) in the BCC-algebra \(X\) is uniformly continuous.

**Proof.** For any \(\varepsilon > 0\), if \(d^*_p((x, y), (a, b)) < \frac{\varepsilon}{2}\), then \(d_p(x, a) < \frac{\varepsilon}{2}\) and \(d_p(y, b) < \frac{\varepsilon}{2}\). Using Proposition 3.33, we have

\[
d_p(x \ast y, y \ast a) \leq d_p((x, y), (a, y)) + d_p(a \ast y, a \ast b) \\
\leq d_p(x, a) + d_p(y, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore the operation \(* : X \times X \to X\) is uniformly continuous. 

Table 3: ∗-operation

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

The following example illustrates Theorem 3.38.

**Example 3.39.** Let $X = \{0, a, b, c\}$ be a set with the ∗-operation given by Table 3. Then $(X, ∗, 0)$ is a proper BCC-algebra. Let $ϕ$ be a real-valued function on $X$ defined by

$$ϕ = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 4 & 5 \end{pmatrix}.$$ 

Then $ϕ$ is a BCK-valuation on $X$ and $(X, d_ϕ)$ is a metric space where

$$d_ϕ = \begin{pmatrix} (0, 0) & (0, a) & (0, b) & (0, c) & (a, a) & (a, b) & (a, c) & (b, b) & (b, c) & (c, c) \\ 0 & 3 & 4 & 5 & 0 & 3 & 4 & 0 & 4 & 0 \end{pmatrix}.$$ 

Also, $(X \times X, d_ϕ^\ast)$ is a metric space where $d_ϕ^\ast$ is obtained by (10), for example,

$$d_ϕ^\ast((0, b), (a, c)) = \max\{d_ϕ(0, a), d_ϕ(b, c)\} = \max\{3, 4\} = 4,$$
$$d_ϕ^\ast((a, b), (c, a)) = \max\{d_ϕ(a, c), d_ϕ(b, a)\} = \max\{4, 3\} = 4,$$
$$d_ϕ^\ast((c, a), (0, 0)) = \max\{d_ϕ(c, 0), d_ϕ(a, 0)\} = \max\{5, 3\} = 5,$$
$$d_ϕ^\ast((a, c), (b, 0)) = \max\{d_ϕ(a, b), d_ϕ(c, 0)\} = \max\{3, 5\} = 5,$$
$$d_ϕ^\ast((a, c), (b, c)) = \max\{d_ϕ(a, b), d_ϕ(c, c)\} = \max\{3, 0\} = 3,$$

and so on. Now, it is routine to verify that the operation ∗ in the BCC-algebra $X$

$$*: X \times X \to X, \quad (x, y) \mapsto x * y$$

is uniformly continuous.

4. Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions.

References