On the torsion graph and von Neumann regular rings

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Abstract. Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. A torsion graph of $M$, denoted by $\Gamma(M)$, is a graph whose vertices are the non-zero torsion elements of $M$, and two distinct vertices $x$ and $y$ are adjacent if and only if $[x : M][y : M]M = 0$. In this paper, we investigate the relationship between the diameters of $\Gamma(M)$ and $\Gamma(R)$, and give some properties of minimal prime submodules of a multiplication $R$-module $M$ over a von Neumann regular ring. In particular, we show that for a multiplication $R$-module $M$ over a Bézout ring $R$ the diameter of $\Gamma(M)$ and $\Gamma(R)$ is equal, where $M \neq T(M)$. Also, we prove that, for a faithful multiplication $R$-module $M$ with $|M| \neq 4$, $\Gamma(M)$ is a complete graph if and only if $\Gamma(R)$ is a complete graph.

1. Introduction

In 1999 Anderson and Livingston [1], introduced and studied the zero-divisor graph of a commutative ring with identity whose vertices are nonzero zero-divisors while $x - y$ is an edge whenever $xy = 0$. Since then, the concept of zero-divisor graphs has been studied extensively by many authors including Badawi and Anderson [7], Anderson, Levy and Shapiro [2] and Mulay [17]. This concept has also been introduced and studied for near-rings, semigroups, and non-commutative rings by Cannon, Neuerburg and Redmond [9], DeMeyer, McKenzie and Schneider [10] and Redmond [18]. For recent developments on graphs of commutative rings see Anderson and Badawi [4], and Anderson, Axtell and Stickles [5].

In 2009, the concept of the zero-divisor graph for a ring has been extended to module theory by Ghalandarzadeh and Malakooti Rad [12]. They defined the torsion graph of an $R$-module $M$ whose vertices are the nonzero torsion elements of $M$ such that two distinct vertices $x$ and $y$ are adjacent if and only if $[x : M][y : M]M = 0$. For a multiplication $R$-module $M$, they proved that, $\Gamma(M)$ and $\Gamma(S^{-1}M)$ are isomorphic, where $S = R \setminus Z(M)$. Also, they showed that, $\Gamma(M)$ is connected and $\text{diam}(\Gamma(M)) \leq 3$ for a faithful $R$-module $M$, see [13].

Let $R$ be a commutative ring with identity and $M$ be a unitary multiplication $R$-module. In this paper, we will investigate the concept of a torsion graph and minimal prime submodules of an $R$-module. Also, we study the relationship among the diameters of $\Gamma(M)$ and $\Gamma(R)$, and minimal prime submodules of a multiplication $R$-module $M$ over a von Neumann regular ring. In particular, we show that for a multiplication $R$-module $M$ over a Bézout ring $R$ the diameter of $\Gamma(M)$ and $\Gamma(R)$ is equal, where $M \neq T(M)$.

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Also, we prove that, if \( \Gamma(M) \) is a complete graph, then \( \Gamma(R) \) is a complete graph for a multiplication \( R \)-module \( M \) with \( |M| \neq 4 \). The converse is true if we assume further that \( M \) is faithful.

An element \( m \) of \( M \) is called a torsion element if and only if it has a non-zero annihilator in \( R \). Let \( T(M) \) be the set of torsion elements of \( M \). It is clear that if \( R \) is an integral domain, then \( T(M) \) is a submodule of \( M \), which is called a torsion submodule of \( M \). If \( T(M) = 0 \), then the module \( M \) is said to be torsion-free, and it is called a torsion module if \( T(M) = M \). Thus, \( \Gamma(M) \) is an empty graph if and only if \( M \) is a torsion-free \( R \)-module. An \( R \)-module \( M \) is called a multiplication \( R \)-module if for every submodule \( N \) of \( M \), there exists an ideal \( I \) of \( R \) such that \( N = IM \), Barnard [8]. Also, a ring \( R \) is called reduced if \( \text{Nil}(R) = 0 \), and an \( R \)-module \( M \) is called a reduced module if \( rm = 0 \) implies that \( rM \cap Rm = 0 \), where \( r \in R \) and \( m \in M \). It is clear that \( M \) is a reduced module if \( r^2m = 0 \) for \( r \in R \), \( m \in M \) implies that \( rm = 0 \). Also by the proof of Lemma 3.7, step 1, in Ghalandarzadeh and Malakooti Rad [12], we can check that a multiplication \( R \)-module \( M \) is reduced if and only if \( \text{Nil}(M) = 0 \). Also, a ring \( R \) is a von Neumann regular ring if for each \( a \in R \), there exists an element \( b \in R \) such that \( a = ab\bar{b} \). It is clear that every von Neumann regular ring is reduced. A submodule \( N \) of an \( R \)-module \( M \) is called a pure submodule of \( M \) if \( IM \cap N = IN \) for every ideal \( I \) of \( R \) Ribenboim [19]. Following Kash ([14], p. 105), an \( R \)-module \( M \) is called a von Neumann regular module if and only if every cyclic submodule of \( M \) is a direct summand in \( M \). If \( N \) is a direct summand in \( M \), then \( N \) is pure but not conversely Matsumara ([16], Example 2, p. 54) and Ribenboim ([19], Example 14, p. 100). And so every von Neumann regular module is reduced. A proper submodule \( N \) of \( M \) is called a prime submodule of \( M \), whenever \( rm \in N \) implies that \( m \in N \) or \( r \in [N : M] \), where \( r \in R \) and \( m \in M \). Also, a prime submodule \( N \) of \( M \) is called a minimal prime submodule of a submodule \( H \) of \( M \), if it contains \( H \) and there is no smaller prime submodule with this property. A minimal prime submodule of the zero submodule is also known as a minimal prime submodule of the module \( M \). Recall that a ring \( R \) is called Bézout if every finitely generated ideal \( I \) of \( R \) is principal. We know that every von Neumann regular ring is Bézout.

A \( G \) is connected if there is a path between any two distinct vertices. The distance \( d(x, y) \) between connected vertices \( x \) and \( y \) is the length of a shortest path from \( x \) to \( y \) \( (d(x, y) = \infty \) if there is no such path). The diameter of \( G \) is the diameter of a connected graph, which is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex. Also, a complete graph is a simple graph whose vertices are pairwise adjacent; the complete graph with \( n \) vertices is denoted by \( K_n \).

Throughout, \( R \) is a commutative ring with identity and \( M \) is a unitary \( R \)-module. The symbol \( \text{Nil}(R) \) will be the ideal consisting of nilpotent elements of \( R \). In addition, \( \text{Spec}(M) \) and \( \text{Min}(M) \) will denote the set of the prime submodules of \( M \) and minimal prime submodules of \( M \), respectively. And \( \text{Nil}(M) := \cap_{N \in \text{Spec}(M)} N \) will denote the nilradical of \( M \). We shall often use \( [x : M] \) and \( [0 : M] = \text{Ann}(M) \) to denote the residual of \( Rx \) by \( M \) and the annihilator of an \( R \)-module \( M \), respectively. The set \( Z(M) := \{ r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M \} \) will denote the zero-divisors of \( M \). As usual, the rings of integers and integers modulo \( n \) will be denoted by \( \mathbb{Z} \) and \( \mathbb{Z}_n \), respectively.

2. Minimal prime submodules

In this section, we investigate some properties of the class of minimal prime submodules of a multiplication \( R \)-module \( M \). Multiplication \( R \)-modules have been studied in El-Bast and Smith [11]. In the mentioned paper they have proved the following theorem.

**Theorem 2.1.** Let \( M \) be a non-zero multiplication \( R \)-module. Then

(1) every proper submodule of \( M \) is contained in a maximal submodule of \( M \), and

(2) \( K \) is maximal submodule of \( M \) if and only if there exists a maximal ideal \( P \) of \( R \) such that \( K = PM \neq M \).

**Proof.** El-Bast and Smith (Theorem 2.5, [11]). \( \Box \)

A consequence of the above theorem is that every non-zero multiplication \( R \)-module has a maximal submodule, since 0 is a proper submodule of \( M \). Therefore every non-zero multiplication \( R \)-module has a prime submodule.
Lemma 2.2. Let \( M \) be a multiplication \( R \)-module. Suppose that \( S \) be a non empty multiplicatively closed subset of \( R \), and \( H \) be a proper submodule of \( M \) such that \([H : M]\) does not meet \( S \). Then there exists a prime submodule \( N \) of \( M \) which contains \( H \) and \([N : M] \cap S = \emptyset\).

Proof. Let \( S \) be a non empty multiplicatively closed subset of \( R \) and \( H \) be a proper submodule of \( M \) such that \([H : M]\) does not meet \( S \). Set

\[
\Omega := \{(K : M)|K < M, [H : M] \subseteq [K : M], [K : M] \cap S = \emptyset\}.
\]

Since \([H : M] \in \Omega\), we have \( \Omega \neq \emptyset \). Of course, the relation of inclusion, \( \subseteq \), is a a partial order on \( \Omega \). Let \( \Delta \) be a non-empty totally ordered subset of \( \Omega \) and \( G = \bigcup_{[K : M]} [K : M] \). It is clear that \( G \in \Omega \); then by Zorn’s Lemma \( \Omega \) has a maximal element say \([N : M]\). We show that \( N = [N : M]M \in \text{Spec}(M) \). Assume \( rm \in N \) for some \( r \in R \) and \( m \in M \), but neither \( r \in [N : M] \) nor \( m \in N \). Hence \( rm \notin N \), and so there is \( m_0 \in M \) such that \( rm_0 \notin N \). Therefore \( N \subset H_1 = rm_0 + N \), and \( N \subset H_2 = Rm + N \). Hence \([N : M] \subset [H_1 : M] \) and \([N : M] \subset [H_2 : M] \). Consequently \([H_1 : M] \) and \([H_2 : M] \) are not elements of \( \Omega \). So \([H_1 : M] \cap S \neq \emptyset \) and \([H_2 : M] \cap S \neq \emptyset \). Thus there are two elements \( s_1, s_2 \in S \) such that \( s_1 M \subseteq H_1 \) and \( s_2 M \subseteq H_2 \). Hence

\[
s_2 s_1 M \subseteq s_2 H_1 \subseteq s_2(Rm_0 + N),
\]

so

\[
s_2 s_1 M \subseteq Rrs_2 m_0 + s_2 N \subseteq R(rM + N) + N \subseteq N.
\]

Therefore \( s_2 s_1 \in [N : M] \cap S \), and we have derived the required contradiction. Consequently \( N \) is a prime submodule of \( M \). \( \square \)

Lemma 2.3. Let \( M \) be an \( R \)-module with \( \text{Spec}(M) \neq \emptyset \), and \( H \) be a submodule of \( M \). Let \( H \) be contained in a prime submodule \( N \) of \( M \), then \( N \) contains a minimal prime submodule of \( H \).

Proof. Suppose that \( \Omega = \{K|K \in \text{Spec}(M), H \subseteq K \subseteq N\} \). Clearly \( N \in \Omega \), and so \( \Omega \) is not empty. If \( N' \) and \( N'' \) belong to \( \Omega \), then we shall write \( N' \leq N'' \) if \( N'' \subseteq N' \). This gives a partial order on \( \Omega \). Now by Zorn’s Lemma \( \Omega \) has a maximal element, say \( N' \). Since \( N' \in \Omega \), \( N' \) is a prime submodule of \( M \). We show that \( N' \) is a minimal prime submodule of \( H \). Let \( H \subseteq N_1 \subseteq N' \). So \( N_1 \subseteq N' \), and since \( N' \) is a maximal in \( \Omega \), \( N' = N_1 \). Consequently \( N' \) is minimal with \( H \subseteq N' \subseteq N \). \( \square \)

Theorem 2.4. Let \( M \) be a multiplication \( R \)-module. Then \( \text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N \).

Proof. Clearly \( \text{Nil}(M) \subseteq \bigcap_{N \in \text{Min}(M)} N \). To establish the reverse inclusion, let \( x \notin \text{Nil}(M) \). We show that there is a minimal prime submodule which does not contain \( x \). Since \( x \notin \text{Nil}(M) \), there is a prime submodule \( N \) of \( M \) such that \( x \notin N \). If for all \( 0 \neq a \in [x : M] \) there exists \( n \in N \) such that \( a^n x = 0 \), then \( x \notin N \); which is a contradiction. Thus there exists non-zero element \( a \in [x : M] \) such that \( a^n x \neq 0 \) for all \( n \in N \). Let \( S = \{a^n|n \geq 0\} \). It is clear that \( S \) is a multiplicatively closed subset of \( R \), and \( 0 \notin S \). A simple check yields that \( S \cap [0 : M] = \emptyset \). By Lemma 2.2, there exists a prime submodule \( N \) of \( M \) such that \( 0 \subseteq N \) and \([N : M] \cap S = \emptyset \). Therefore by Lemma 2.3, there exists a minimal prime submodule \( N' \) of \( M \) such that \( 0 \subseteq N' \subseteq N \). Since \( x \notin N \), we have \( x \notin N' \). Consequently \( \text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N \). \( \square \)

Lemma 2.5. Let \( R \) be a von Neumann regular ring. Then every \( R \)-module is reduced.

Proof. Let \( R \) be a von Neumann regular ring. So any finitely generated ideal is generated by an idempotent, and therefore any \( R \)-module is reduced. \( \square \)

Proposition 2.6. Let \( R \) be a von Neumann regular ring, and \( M \) be a multiplication \( R \)-module. Suppose that \( \Gamma(M) \) be a connected graph, and \( \Gamma(M) \neq K_1 \). Then \( T(M) = \bigcup_{N \in \text{Min}(M)} N \).
Proof. Let $N$ be a prime submodule of $M$ such that $N \not\subseteq T(M)$. It will be sufficient to show that $N \notin \text{Min}(M)$. Since $N \not\subseteq T(M)$, we may suppose that there exists an element $x \in N$ such that $x \notin T(M)$. Since $M$ is a multiplication module, we may assume $x = \sum_{i=1}^n a_i m_i$. Since $R$ is a von Neumann regular ring, we have $\sum_{i=1}^n R a_i = Re$ for some non-zero idempotent element $e$ of $R$. Therefore there exists $m \in M$ such that $x = em$. Now put $\Omega = \{e^2 \beta | 0 = 1 \text{ and } \beta \in R \setminus \{N : M\}\}$. Since $x = em \notin T(M)$, we have $R \setminus \{N : M\} \subseteq \Omega$, and $0 \notin \Omega$. Now a simple check shows that $\Omega$ and $R \setminus \{N : M\}$ are multiplicatively closed subsets of $R$. Let $\Delta = \{S|S$ is a multiplicatively closed subset of $R\}$. $R \setminus \{N : M\}$ is not a maximal element of $\Delta$, since $R \setminus \{N : M\} \subseteq \Omega$. Thus $\{N : M\}$ is not a minimal prime ideal of $R$, and so there exists a prime ideal $h_1$ of $R$ such that $h_1 M \subset N$. Therefore $h_1 M \cap N = \{N : M\}$, and so $N \notin \text{Min}(M)$. Accordingly $\bigcup_{\text{Min}(M) \not\subset N} N \subseteq T(M)$.

Now let $x \in T(M)^* \setminus x \notin \bigcup_{\text{Min}(M) \not\subset N} N$. Therefore $x \notin N$ for all minimal prime submodules $N$ of $M$. Since $\Gamma(M)$ is connected and $\Gamma(M) \neq K_1$, there is $y \in T(M)$ such that $x \neq y$ and $[x : M][y : M] = 0$ and so $\text{Ann}(x) \neq \text{Ann}(M)$. So there is a non-zero element $r \in \text{Ann}(x)$ such that $r \notin \text{Ann}(M)$. Thus $rx = 0 \in N$ for all minimal prime submodules $N$ of $M$. Since $x \notin N$, then $rM \subseteq \bigcap_{\text{Min}(M) \not\subset N} N$. Now by Theorem 2.4, $rM \subseteq \text{Nil}(M)$ and since $R$ is a von Neumann regular ring, by Lemma 2.5, $M$ is a reduced module and $\text{Nil}(M) = 0$. Hence $r \in \text{Ann}(M)$, which is a contradiction. Therefore, $x \notin \bigcup_{\text{Min}(M) \not\subset N} N$. □

The next result give some properties and characterizations of multiplication von Neumann regular modules as a generalization of von Neumann regular rings.

**Proposition 2.7.** Let $M$ be a multiplication $R$-module.

1. If $R$ be a von Neumann regular ring, then $M$ is a von Neumann regular module.

2. If $R$ be a von Neumann regular ring, then $S^{-1}M$ is a von Neumann regular module, and $\text{Nil}(S^{-1}M) = 0$, where $S = R \setminus Z(M)$.

Proof. (1) Let $0 \neq x = \sum_{i=1}^n a_i M_i \in M$, where $a_i \in [x : M], M_i \in M$. Since $R$ is a von Neumann regular ring, we have $\sum_{i=1}^n R a_i = Re$ for some non-zero idempotent element $e$ of $R$; therefore there exists $m \in M$ such that $x = em$ and $e \in [x : M]$. So $1 = e + 1 - e$, thus

$$M = eM + (1 - e)M \subseteq Rx + M(1 - e).$$

Now, let $y \in Rx \cap M(1 - e)$. Hence $y = r_1 x = (1 - e)m$ for some $r_1 \in R$ and $m \in M$; so $y = ey = r_1 em_1 = e(1 - e)m = 0$. Therefore $M = Rx \oplus M(1 - e)$ and $M$ is a von Neumann regular module.

(2) We show that $SM = M$ for all $s \in S$, where $S = R \setminus Z(M)$. Since $R$ is a von Neumann regular ring, for any $s \in S$ there exists $t \in S$ such that $s + t = u$ is a regular element of $R$ and $st = 0$. So $u$ is a unit of $R$; hence $uM = M$. Since $st = 0$ and $s \notin Z(M), tM = 0$. Therefore $M = sM$ for all $s \in S$. Thus $S^{-1}M = M$. By (1), $S^{-1}M$ is a von Neumann regular module. □

3. The diameter of torsion graphs

In this section we establish some basic and important results on the diameter of torsion graphs over a multiplication module. Moreover, we investigate the relationship between the diameter of $\Gamma(M)$ and $\Gamma(R)$.

**Theorem 3.1.** Let $M$ be a multiplication $R$-module with $|M| \neq 4$. If $\Gamma(M)$ is a complete graph, then $\Gamma(R)$ is a complete graph. The converse is true if we assume further that $M$ is faithful.

Proof. Let $\Gamma(M)$ be a complete graph. By Ghalandarzadeh and Malakooti (Theorem 2.11, [13]), $\text{Nil}(M) = T(M)$. Also by Theorem 2.4, $\text{Nil}(M) = \bigcap_{\text{Min}(M)} N$, so $T(M) \neq M$. Hence there exists $m \in M$ such that $\text{Ann}(m) = 0$. Suppose that $\alpha, \beta$ are two vertices of $\Gamma(R)$. One can easily check that $\alpha m, \beta m \in T(M)^*$. Therefore $[\alpha m : M][\beta m : M]M = 0$, so $\alpha^2 g = 0$. Consequently $\Gamma(R)$ is a complete graph.

Now, let $\Gamma(R)$ be a complete graph, and $m, n \in T(M)^*$. So $\text{Ann}(m) = 0$ and $\text{Ann}(n) = 0$. Suppose that $0 \neq \alpha \in [m : M]$ and $0 \neq \beta \in [n : M]$. Since $M$ is a faithful, $R$-module then $\alpha$ and $\beta$ are the vertices $\Gamma(R)$. Therefore $\alpha \beta = 0$, and so $[m : M][n : M]M = 0$. Hence $\Gamma(M)$ is a complete graph. □
The following example shows that the multiplication condition in the above theorem is not superfluous.

Example 3.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}_6$. So by El-Bast and Smith (Corollary 2.3, [11]), $M$ is not a multiplication $R$-module. Also $\Gamma(M)$ is a complete graph, but $V(\Gamma(R)) = \emptyset$.

Corollary 3.3. Let $M$ be a faithful multiplication $R$-module with $|M| \neq 4$. If $\Gamma(R)$ is a complete graph, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\text{Nil}(M) = \text{Nil}(R)M = Z(R)M = T(M)$.

Proof. Let $\Gamma(M)$ be a faithful multiplication $R$-module. By Theorem 3.1, $\Gamma(M)$ is a complete graph, and by Ghalandarzadeh and Malakooti (Theorem 2.11, [13]), $\text{Nil}(M) = T(M)$. Let $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$, by Anderson and Livingston (Theorem 2.8, [1]), $\text{Nil}(R) = Z(R)$. Hence $Z(R)$ is an ideal of $R$ and $T(M) = Z(R)M$. Therefore, we have that $\text{Nil}(M) = \text{Nil}(R)M = Z(R)M = T(M)$. □

Corollary 3.4. Let $M$ be a faithful multiplication $R$-module with $|M| \neq 4$. If $\Gamma(R)$ is a complete graph, then $|\text{Min}(M)| = 1$.

Proof. Let $M$ be a faithful multiplication $R$-module. By Theorem 3.1, $\Gamma(M)$ is a complete graph. Thus $T(M)$ is a submodule of $M$. We show that $\bigcup_{N \subseteq \text{Min}(M)} N \subseteq T(M)$. Suppose that $N$ be a prime submodule of $M$, such that $N \not\subseteq T(M)$. It will be sufficient to show that $N \not\subseteq \text{Min}(M)$. Since $N \not\subseteq T(M)$ there exists an element $x \in N$ such that $x \notin T(M)$. So there are $a \in [x : M]$ and $m \in M$ such that $am \notin T(M)$. Now by putting $\Omega = \{a^i \beta^j \mid i \geq 0 \text{ and } \beta \in R \setminus [N : M]\}$, and similar to the proof of Proposition 2.6, one can check that $\bigcup_{N \subseteq \text{Min}(M)} N \subseteq T(M)$. By Ghalandarzadeh and Malakooti (Theorem 2.11, [13]) and Theorem 2.4, we have

$$
\bigcup_{N \subseteq \text{Min}(M)} N \subseteq T(M) = \text{Nil}(M) = \bigcap_{N \subseteq \text{Min}(M)} N,
$$

which completes the proof. □

Theorem 3.5. Let $R$ be a Bézout ring and $M$ be a multiplication $R$-module such that $|M| \neq 4$ and $M \neq T(M)$; then $\text{diam}(\Gamma(M)) = \text{diam}(\Gamma(R))$.

Proof. Let $R$ be a Bézout ring, and $M$ be a multiplication $R$-module. By Theorem 3.1, $\text{diam}(\Gamma(M)) = 1$ if and only if $\text{diam}(\Gamma(R)) = 1$. Suppose that $\text{diam}(\Gamma(R)) = 1$ and $x, y \in T(M)^\prime$ such that $d(x, y) \neq 1$. Let $x = \sum_{i=1}^m a_im_i$ and $y = \sum_{i=1}^m b_im_i$, where $0 \neq a_i \in [x : M], 0 \neq b_i \in [y : M]$. Since $R$ is a Bézout ring, $\sum_{i=1}^m R \alpha_i = Ra$ and $\sum_{i=1}^m R \beta_i = R\beta$, for some $\alpha, \beta \in R$. Hence there exist $m_i, m_j \in M$ such that $x = am_i, y = \beta m_j$. Thus $\alpha, \beta \in Z(R)$. If $d(\alpha, \beta) = 1$, then $d(x, y) = 1$, and so we have a contradiction. Thus $d(\alpha, \beta) = 2$, so there exists $x \in Z(R)^\prime$ such that $\alpha - \gamma - \beta$ is a path of length 2. Since $M \neq T(M)$, then there is $n \in M$ such that $\gamma n \in T(M)^\prime$. Therefore $an = x - \gamma n - y = \beta m_i$. is a path of length 2. So $d(x, y) = 2$ and $\text{diam}(\Gamma(M)) = 2$.

Suppose that $\text{diam}(\Gamma(M)) = 2$ and $\alpha, \beta \in Z(R)$ such that $d(\alpha, \beta) \neq 1$. So $\alpha \beta \neq 0$; since $M \neq T(M)$, then there is $n \in M$ such that $\alpha \beta n \neq 0$. Hence $\beta n \neq \alpha n \in T(M)^\prime$. If $d(\alpha \beta n, \beta n) = 1$, then $[\alpha n : M] \beta n = 0$. So $\alpha \beta n = 0$, which is a contradiction. So $d(\alpha \beta n, \beta n) = 2$, and there is $x \in [\alpha \gamma n \in T(M)^\prime$ such that $\alpha n - \gamma n - \beta n$, is a path of length 2. Thus $[\alpha n : M] \gamma n = 0 = [\beta n : M] \gamma n$, so $\alpha \gamma n = \beta \gamma$ and $\alpha - \gamma - \beta$ is a path of length 2. Therefore $\text{diam}(\Gamma(R)) = 2$.

Now, let $\text{diam}(\Gamma(R)) = 3$, so $\text{diam}(\Gamma(M)) \geq 3$, and by Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\text{diam}(\Gamma(M)) \leq 3$. Therefore $\text{diam}(\Gamma(M)) = 3$. If $\text{diam}(\Gamma(M)) = 3$, then $\text{diam}(\Gamma(R)) \geq 3$, and by Anderson and Livingston, (Theorem 2.3, [1]), $\text{diam}(\Gamma(R)) \leq 3$. Therefore $\text{diam}(\Gamma(R)) = 3$. Consequently $\text{diam}(\Gamma(M)) = \text{diam}(\Gamma(R))$. □
Lemma 3.6. Let $M$ be a reduced multiplication $R$-module and $H$ be a finitely generated submodule of $M$. Then $\text{Ann}(H)M \neq 0$ if and only if $H \subseteq N$ for some $N \in \text{Min}(M)$.

Proof. Let $\text{Ann}(H)M \neq 0$, so $\text{Ann}(H)M \not\subseteq \text{Nil}(M) = \bigcap_{N \in \text{Min}(M)} N$. Thus there exists $N_0 \in \text{Min}(M)$ such that $\text{Ann}(H)M \not\subseteq N_0$. Assume that $r \in R$ and $m \in M$ and $rm \in \text{Ann}(H)M$, but $rm \not\in N_0$. Therefore $rm[H : M] = 0 \subseteq N_0$. Since $rm \not\in N_0$, we have $H \subseteq N_0$.

To establish the reverse, let $P = PM \in \text{Min}(M)$, where $P = [N : M]$, and $H \subseteq N$. Since $M$ is a reduced $R$-module, $M_P$ will be a reduced $R_P$-module. We show that $M_P$ has exactly one maximal submodule. Let $M_P$ have two maximal submodules $S^{-1}H_1$ and $S^{-1}H_2$; so there exist two ideals $S^{-1}h_1$ and $S^{-1}h_2$ of $\text{Max}(S^{-1}R)$, such that $S^{-1}H_1 = S^{-1}h_1 S^{-1}M$ and $S^{-1}H_2 = S^{-1}h_2 S^{-1}M$. Since $R_P$ is a local ring, $S^{-1}H_1 = S^{-1}H_2$. We know that $S^{-1}N$ is a proper submodule of $S^{-1}M$, and so by Theorem 2.1, $S^{-1}PS^{-1}M = S^{-1}N$ is the unique maximal submodule of $M_P$. Also, if $S^{-1}H_0$ is a prime submodule of $M_P$, then by Theorem 2.1, $S^{-1}H_0 \subseteq S^{-1}N$. By a routine argument $H_0 \subseteq N$, so $H_0 = N$; hence $S^{-1}H_0 = S^{-1}N$. Therefore by Theorem 2.4, $\text{Nil}(M_P) = S^{-1}N$. Since $M_P$ is reduced, $\text{Nil}(M_P) = 0$. Thus $S^{-1}H_0 = 0$. Suppose that $H = \sum_{i=1}^{n} Rh_i$, so $\frac{1}{S} = 0$ for all $1 \leq i \leq n$. Hence there exists $s_i \in R \setminus P$ such that $s_i h_i = 0$. Let $s = s_1 s_2 \cdots s_n$, thus $sh = 0$. If $sM = 0$ then $s \in [N : M] = P$, which is a contradiction. So there is an element $m \in M$ such that $0 \neq sm \in \text{Ann}(H)M$.  

Theorem 2.6 in [15] characterizes the diameter of $\Gamma(R)$ in terms of the ideals of $R$. Our results obtained in Theorems 3.7 and 3.8 specifies the diameter of $\Gamma(M)$ in terms of minimal prime submodules of a multiplication module $M$ over a von Neumann regular ring.

Theorem 3.7. Let $R$ be a von Neumann regular ring and $M$ be a multiplication $R$-module. If $M$ has more than two minimal prime submodules and $T(M)$ is not a submodule of $M$, then $\text{diam}(\Gamma(M)) = 3$.

Proof. Let $m, n$ be two distinct elements of $T(M)'$ and $\text{Ann}(Rm + Rn) = 0$. Hence $M$ is faithful. First, suppose that $[m : M][n : M]M \neq 0$, so $d(m, n) \neq 1$. If $d(m, n) = 2$, then there exists a vertex $x \in T(M)'$ such that $m - x - n$ is a path. Thus $[m : M][x : M]M = 0 = [x : M][n : M]M$.

Accordingly $[x : M][Rm + Rn] = 0$, and so $[x : M] \not\subseteq \text{Ann}(Rm + Rn) = 0$. Which is a contradiction. We shall now assume that $d(m, n) \neq 2$. By Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\Gamma(M)$ is connected with $\text{diam}(\Gamma(M)) \leq 3$; therefore $d(m, n) = 3$. Next, assume $[m : M][n : M]M \neq 0$, then by Proposition 2.6, $m, n \in \bigcup_{N \in \text{Min}(M)} N$. Since $\text{Ann}(Rm + Rn)M = 0$, by Lemma 3.6, $m$ and $n$ belong to two distinct minimal prime submodules. Suppose that $P, N$ and $Q$ are distinct minimal prime submodules of $M$ such that $m \in P \setminus (Q \cup N)$ and $n \in (Q \cap N) \setminus P$. Hence $[m : M]M \not\subseteq N$; thus $an \not\in N$ for some $a \in [m : M]$ and $m \in M$. Let $x \in (Q \cap P) \setminus N$. A simple check yields that $a^2 x = 0$. On the other hand, since $[m : M][n : M]M = 0$, we have $a(n + ax) = a^2 x$. Therefore $0 \neq a^2 x \in [m : M][n + ax : M]M$. Also, by a routine argument, we have $Rm + Rn = Rm + R(n + ax)$.

So $\text{Ann}(Rm + R(n + ax)) = 0$. Similar to the above argument, we have $d(m, (n + ax)) = 3$. Consequently $\text{diam}(\Gamma(M)) = 3$.  

Theorem 3.8. Let $R$ be a von Neumann regular ring and $M$ be a multiplication $R$-module. If $T(M)$ is not a submodule of $M$, then $\text{diam}(\Gamma(M)) \leq 2$ if and only if $M$ has exactly two minimal prime submodules.

Proof. Suppose that $\text{diam}(\Gamma(M)) \leq 2$, and $T(M)$ is not a submodule of $M$, so there exist $m, n \in T(M)'$ with $\text{Ann}(Rm + Rn) = 0$. So $M$ is faithful and by Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\Gamma(M)$ is connected. Now since $\Gamma(M)$ is a connected graph and $T(M)$ is not a submodule of $M$, by Proposition 2.6 and Lemma 3.6, there are at least two distinct minimal prime submodules $P$ and $Q$ of $M$ such that $m \in P \setminus Q$ and $n \in Q \setminus P$. On the other hand, by Theorem 3.7, $M$ can not have more than two minimal prime submodules; therefore $M$ has exactly two minimal prime submodules. Conversely, suppose that $P$ and $Q$ be only two minimal prime submodules of $M$. By Proposition 2.6, $T(M) = P \cup Q$. Assume that $m, n \in T(M)'$ such that $m \in P \setminus Q$ and $n \in Q \setminus P$. Thus $[m : M][n : M]M \subseteq P \cap Q = \text{Nil}(M) = 0$, by Lemma 2.5. So $d(m, n) = 1$. Also if $m, n \in P$, then $Rm + Rn \subseteq P$. By Lemma 3.6, $\text{Ann}(Rm + Rn)M \neq 0$; therefore there is $0 \neq a \in R$ such that $am = an = 0$. On the other hand, there exists a non-zero element $x$ of $M$ such that $ax \neq 0$ and so $m - ax - n$ is a path, hence $d(m, n) = 2$, thus $\text{diam}(\Gamma(M)) \leq 2$. Moreover, if $n, m \in Q$, then similarly $\text{diam}(\Gamma(M)) \leq 2$.  

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As an immediate consequence from Theorem 3.5 and Theorem 3.8, we obtain the following result.

**Corollary 3.9.** Let $R$ be a von Neumann regular ring and let $M$ be a multiplication $R$-module. If $T(M)$ is not a submodule of $M$, then $M$ has exactly two minimal prime submodules if and only if $R$ has exactly two minimal prime ideals.

**References**