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On the torsion graph and von Neumann regular rings

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. A torsion graph of M, denoted by $\Gamma(M)$, is a graph whose vertices are the non-zero torsion elements of M, and two distinct vertices x and y are adjacent if and only if $[x : M][y : M]M = 0$. In this paper, we investigate the relationship between the diameters of $\Gamma(M)$ and $\Gamma(R)$, and give some properties of minimal prime submodules of a multiplication R-module M over a von Neumann regular ring. In particular, we show that for a multiplication R-module M over a Bézout ring R the diameter of $\Gamma(M)$ and $\Gamma(R)$ is equal, where $M \neq T(M)$. Also, we prove that, for a faithful multiplication R-module M with $|M| \neq 4$, $\Gamma(M)$ is a complete graph if and only if $\Gamma(R)$ is a complete graph.

1. Introduction

In 1999 Anderson and Livingston [1], introduced and studied the zero-divisor graph of a commutative ring with identity whose vertices are nonzero zero-divisors while $x-y$ is an edge whenever $xy = 0$. Since then, the concept of zero-divisor graphs has been studied extensively by many authors including Badawi and Anderson [7], Anderson, Levy and Shapiro [2] and Mulay [17]. This concept has also been introduced and studied for near-rings, semigroups, and non-commutative rings by Cannon, Neuerburg and Redmond [9], DeMeyer, McKenzie and Schneider [10] and Redmond [18]. For recent developments on graphs of commutative rings see Anderson and Badawi [4], and Anderson, Axtell and Stickles [5].

In 2009, the concept of the zero-divisor graph for a ring has been extended to module theory by Ghalandarzadeh and Malakooti Rad [12]. They defined the torsion graph of an R-module M whose vertices are the nonzero torsion elements of M such that two distinct vertices x and y are adjacent if and only if $[x : M][y : M]M = 0$. For a multiplication R-module M, they proved that, $\Gamma(M)$ and $\Gamma(S^{-1}M)$ are isomorphic, where $S = R \setminus Z(M)$. Also, they showed that, $\Gamma(M)$ is connected and $diam(\Gamma(M)) \leq 3$ for a faithful R-module M, see [13].

Let R be a commutative ring with identity and M be a unitary multiplication R-module. In this paper, we will investigate the concept of a torsion graph and minimal prime submodules of an R-module. Also, we study the relationship among the diameters of $\Gamma(M)$ and $\Gamma(R)$, and minimal prime submodules of a multiplication R-module M over a von Neumann regular ring. In particular, we show that for a multiplication R-module M over a Bézout ring R the diameter of $\Gamma(M)$ and $\Gamma(R)$ is equal, where $M \neq T(M)$.

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Also, we prove that, if $\Gamma(M)$ is a complete graph, then $\Gamma(R)$ is a complete graph for a multiplication R-module M with $|M| \neq 4$. The converse is true if we assume further that M is faithful.

An element m of M is called a torsion element if and only if it has a non-zero annihilator in R. Let $T(M)$ be the set of torsion elements of M. It is clear that if R is an integral domain, then $T(M)$ is a submodule of M, which is called a torsion submodule of M. If $T(M) = 0$, then the module M is said to be torsion-free, and it is called a torsion module if $T(M) = M$. Thus, $\Gamma(M)$ is an empty graph if and only if M is a torsion-free R-module. An R-module M is called a multiplication R-module if for every submodule N of M, there exists an ideal I of R such that $N = IM$, Barnard [8]. Also, a ring R is called reduced if $Nil(R) = 0$, and an R-module M is called a reduced module if $rm = 0$ implies that $rM \cap Rm = 0$, where $r \in R$ and $m \in M$. It is clear that M is a reduced module if $r^2m = 0$ for $r \in R$, $m \in M$ implies that $rm = 0$. Also by the proof of Lemma 3.7, step 1, in Ghalandarzadeh and Malakooti Rad [12], we can check that a multiplication R-module M is reduced if and only if $Nil(M) = 0$. Also, a ring R is a von Neumann regular ring if for each $a \in R$, there exists an element $b \in R$ such that $a = a^2b$. It is clear that every von Neumann regular ring is reduced. A submodule N of an R-module M is called a pure submodule of M if $IM \cap N = IN$ for every ideal I of R Ribenboim [19]. Following Kash ([14], p. 105), an R-module M is called a von Neumann regular module if and only if every cyclic submodule of M is a direct summand in M. If N is a direct summand in M, then N is pure but not conversely Matsumara ([16], Example. 2, p. 54) and Ribenboim ([19], Example. 14, p. 100). And so every von Neumann regular module is reduced. A proper submodule N of M is called a prime submodule of M, whenever $rm \in N$ implies that $m \in N$ or $r \in [N : M]$, where $r \in R$ and $m \in M$. Also, a prime submodule N of M is called a minimal prime submodule of a submodule H of M, if it contains H and there is no smaller prime submodule with this property. A minimal prime submodule of the zero submodule is also known as a minimal prime submodule of the module M. Recall that a ring R is called Bézout if every finitely generated ideal I of R is principal. We know that every von Neumann regular ring is Bézout.

A G is connected if there is a path between any two distinct vertices. The distance $d(x, y)$ between connected vertices x and y is the length of a shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of G is the diameter of a connected graph, which is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex. Also, a complete graph is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n.

Throughout, R is a commutative ring with identity and M is a unitary R-module. The symbol $Nil(R)$ will be the ideal consisting of nilpotent elements of R. In addition, $Spec(M)$ and $Min(M)$ will denote the set of the prime submodules of M and minimal prime submodules of M, respectively. And $Nil(M) := \cap_{N \in Spec(M)} N$ will denote the nilradical of M. We shall often use $[x : M]$ and $[0 : M] = Ann(M)$ to denote the residual of Rx by M and the annihilator of an R-module M, respectively. The set $Z(M) := \{r \in R | \ rm = 0$ for some $0 \neq m \in M\}$ will denote the zero-divisors of M. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

2. Minimal prime submodules

In this section, we investigate some properties of the class of minimal prime submodules of a multiplication R-module M. Multiplication R- modules have been studied in El-Bast and Smith [11]. In the mentioned paper they have proved the following theorem.

Theorem 2.1. *Let M be a non-zero multiplication R-module. Then*

(1) every proper submodule of M is contained in a maximal submodule of M, and

(2) K is maximal submodule of M if and only if there exists a maximal ideal P of R such that $K = PM \neq M$.

Proof. El-Bast and Smith (Theorem 2.5, [11]). □

A consequence of the above theorem is that every non-zero multiplication R-module has a maximal submodule, since 0 is a proper submodule of M. Therefore every non-zero multiplication R-module has a prime submodule.

Lemma 2.2. *Let M be a multiplication R-module. Suppose that S be a non empty multiplicatively closed subset of R, and H be a proper submodule of M such that $[H : M]$ dose not meet S. Then there exists a prime submodule N of M which contains H and $[N : M] \cap S = \emptyset$.*

Proof. Let S be a non empty multiplicatively closed subset of R and H be a proper submodule of M such that $[H : M]$ dose not meet S. Set

$$\Omega := \{[K : M] | K < M, [H : M] \subseteq [K : M], [K : M] \cap S = \emptyset\}.$$

Since $[H : M] \in \Omega$, we have $\Omega \neq \emptyset$. Of course, the relation of inclusion, $\subseteq$, is a a partial order on Ω. Let Δ be a non-empty totally ordered subset of Ω and $G = \bigcup_{[K:M]\in\Delta}[K : M]$. It is clear that $G \in \Omega$; then by Zorn's Lemma Ω has a maximal element say $[N : M]$. We show that $N = [N : M]M \in Spec(M)$. Assume $rm \in N$ for some $r \in R$ and $m \in M$, but neither $r \in [N : M]$ nor $m \in N$. Hence $rM \not\subseteq N$, and so there is $m_0 \in M$ such that $rm_0 \notin N$. Therefore $N \subset H_1 = Rrm_0 + N$, and $N \subset H_2 = Rm + N$. Hence $[N : M] \subset [H_1 : M]$ and $[N : M] \subset H_2$. Consequently $[H_1 : M]$ and $[H_2 : M]$ are not elements of Ω. So $[H_1 : M] \cap S \neq \emptyset$ and $[H_2 : M] \cap S \neq \emptyset$. Thus there are two elements $s_1, s_2 \in S$ such that $s_1M \subseteq H_1$ and $s_2M \subseteq H_2$. Hence

$$s_2s_1M \subseteq s_2H_1 \subseteq s_2(Rrm_0 + N),$$

so

$$s_2s_1M \subseteq Rrs_2m_0 + s_2N \subseteq Rr(Rm + N) + N \subseteq N.$$

Therefore $s_2s_1 \in [N : M] \cap S$, and we have derived the required contradiction. Consequently N is a prime submodule of M. □

Lemma 2.3. *Let M be an R-module with $Spec(M) \neq \emptyset$, and H be a submodule of M. Let H be contained in a prime submodule N of M, then N contains a minimal prime submodule of H.*

Proof. Suppose that $\Omega = \{K | K \in Spec(M), H \subseteq K \subseteq N\}$. Clearly $N \in \Omega$, and so Ω is not empty. If N' and N'' belong to Ω, then we shall write $N' \leq N''$ if $N'' \subseteq N'$. This gives a partial order on Ω. Now by Zorn's Lemma Ω has a maximal element, say N^*. Since $N^* \in \Omega$, N^* is a prime submodule of M. We show that N^* is a minimal prime submodule of H. Let $H \subseteq N_1 \subseteq N^*$. So $N^* \leq N_1$, and since N^* is a maximal in Ω, $N^* = N_1$. Consequently N^* is minimal with $H \subseteq N^* \subseteq N$. □

Theorem 2.4. *Let M be a multiplication R-module. Then $Nil(M) = \bigcap_{N\in Min(M)} N$.*

Proof. Clearly $Nil(M) \subseteq \bigcap_{N\in Min(M)} N$. To establish the reverse inclusion, let $x \notin Nil(M)$. We show that there is a minimal prime submodule which dose not contain x. Since $x \notin Nil(M)$, there is a prime submodule N of M such that $x \notin N$. If for all $0 \neq \alpha \in [x : M]$ there exists $n \in \mathbb{N}$ such that $\alpha^n x = 0$, then $x \in N$; which is a contradiction. Thus there exists non-zero element $\alpha \in [x : M]$ such that $\alpha^n x \neq 0$ for all $n \in \mathbb{N}$. Let $S = \{\alpha^n | n \geq 0\}$. It is clear that S is a multiplicatively closed subset of R, and $0 \notin S$. A simple check yields that $S \cap [0 : M] = \emptyset$. By Lemma 2.2, there exists a prime submodule N of M such that $0 \subseteq N$ and $[N : M] \cap S = \emptyset$. Therefore by Lemma 2.3, there exists a minimal prime submodule N^* of M such that $0 \subseteq N^* \subseteq N$. Since $x \notin N$, we have $x \notin N^*$. Consequently $Nil(M) = \bigcap_{N\in Min(M)} N$. □

Lemma 2.5. *Let R be a von Neumann regular ring. Then every R-module is reduced.*

Proof. Let R be a von Neumann regular ring. So any finitely generated ideal is generated by an idempotent, and therefore any R-module is reduced. □

Proposition 2.6. *Let R be a von Neumann regular ring, and M be a multiplication R-module. Suppose that $\Gamma(M)$ be a connected graph, and $\Gamma(M) \neq K_1$. Then $T(M) = \bigcup_{N\in Min(M)} N$.*

Proof. Let N be a prime submodule of M such that $N \nsubseteq T(M)$. It will be sufficient to show that $N \notin Min(M)$. Since $N \nsubseteq T(M)$, we may suppose that there exists an element $x \in N$ such that $x \notin T(M)$. Since M is a multiplication module, we may assume $x = \sum_{i=1}^{n} \alpha_i m_i$. Since R is a von Neumann regular ring, we have $\sum_{i=1}^{n} R\alpha_i = Re$ for some non-zero idempotent element e of R. Therefore there exists $m \in M$ such that $x = em$. Now put $\Omega = \{e^i\beta | i = 0, 1 \text{ and } \beta \in R \setminus [N : M]\}$. Since $x = em \notin T(M)$, we have $R \setminus [N : M] \subset \Omega$, and $0 \notin \Omega$. Now a simple check shows that Ω and $R \setminus [N : M]$ are multiplicatively closed subsets of R. Let $\Delta = \{S | S$ is a multiplicatively closed subset of $R\}$. $R \setminus [N : M]$ is not a maximal element of Δ, Since $R \setminus [N : M] \subset \Omega$. Thus $[N : M]$ is not a minimal prime ideal of R, and so there exists a prime ideal h_1 of R such that $h_1 M \subset N$. Therefore $h_1 M \neq M$ and by El-Bast and Smith (Corollary 2.11, [11]), $h_1 M$ is a prime submodule of M. Therefore $h_1 M \subset N = [N : M]M$, thus $N \notin Min(M)$. Accordingly $\bigcup_{N \in Min(M)} N \subseteq T(M)$.

Now let $x \in T(M)^*$ but, $x \notin \bigcup_{N \in Min(M)} N$. Therefore $x \notin N$ for all minimal prime submodules N of M. Since $\Gamma(M)$ is connected and $\Gamma(M) \neq K_1$, there is $y \in T(M)^*$ such that $x \neq y$ and $[x : M][y : M]M = 0$ and so $Ann(x) \neq Ann(M)$. So there is a non-zero element $r \in Ann(x)$ such that $r \notin Ann(M)$. Thus $rx = 0 \in N$ for all minimal prime submodules N of M. Since $x \notin N$, then $rM \subseteq \bigcap_{N \in Min(M)} N$. Now by Theorem 2.4, $rM \subseteq Nil(M)$ and since R is a von Neumann regular ring, by Lemma 2.5, M is a reduced module and $Nil(M) = 0$. Hence $r \in Ann(M)$, which is a contradiction. Therefore, $x \in \bigcup_{N \in Min(M)} N$. □

The next result give some properties and characterizations of multiplication von Neumann regular modules as a generalization of von Neumann regular rings.

Proposition 2.7. *Let M be a multiplication R-module.*

(1) If R be a von Neumann regular ring, then M is a von Neumann regular module.

(2) If R be a von Neumann regular ring, then $S^{-1}M$ is a von Neumann regular module, and $Nil(S^{-1}M) = 0$, where $S = R \setminus Z(M)$.

Proof. (1) Let $0 \neq x = \sum_{i=1}^{n} \alpha_i m_i \in M$, where $\alpha_i \in [x : M], m_i \in M$. Since R is a von Neumann regular ring, we have $\sum_{i=1}^{n} R\alpha_i = Re$ for some non-zero idempotent element e of R; therefore there exists $m \in M$ such that $x = em$ and $e \in [x : M]$. So $1 = e + 1 - e$, thus

$$M = eM + (1 - e)M \subseteq Rx + M(1 - e).$$

Now, let $y \in Rx \cap M(1 - e)$. Hence $y = r_1 x = (1 - e)m$ for some $r_1 \in R$ and $m \in M$; so $y = ey = r_1 e m_1 = e(1 - e)m = 0$. Therefore $M = Rx \oplus M(1 - e)$ and M is a von Neumann regular module.

(2) We show that $sM = M$ for all $s \in S$, where $S = R \setminus Z(M)$. Since R is a von Neumann regular ring, for any $s \in S$ there exists $t \in S$ such that $s + t = u$ is a regular element of R and $st = 0$. So u is a unit of R; hence $uM = M$. Since $st = 0$ and $s \notin Z(M)$, $tM = 0$. Therefore $M = sM$ for all $s \in S$. Thus $S^{-1}M = M$. By (1), $S^{-1}M$ is a von Neumann regular module. □

3. The diameter of torsion graphs

In this section we establish some basic and important results on the diameter of torsion graphs over a multiplication module. Moreover, we investigate the relationship between the diameter of $\Gamma(M)$ and $\Gamma(R)$.

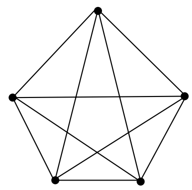
Theorem 3.1. *Let M be a multiplication R-module with $|M| \neq 4$. If $\Gamma(M)$ is a complete graph, then $\Gamma(R)$ is a complete graph. The converse is true if we assume further that M is faithful.*

Proof. Let $\Gamma(M)$ be a complete graph. By Ghalandarzadeh and Malakooti (Theorem 2.11, [13]), $Nil(M) = T(M)$. Also by Theorem 2.4, $Nil(M) = \bigcap_{N \in Min(M)} N$, so $T(M) \neq M$. Hence there exists $m \in M$ such that $Ann(m) = 0$. Suppose that α, β are two vertices of $\Gamma(R)$. One can easily check that $\alpha m, \beta m \in T(M)^*$. Therefore $[\alpha m : M][\beta m : M]M = 0$, so $\alpha\beta = 0$. Consequently $\Gamma(R)$ is a complete graph.

Now, let $\Gamma(R)$ be a complete graph, and $m, n \in T(M)^*$. So $Ann(m) \neq 0$ and $Ann(n) \neq 0$. Suppose that $0 \neq \alpha \in [m : M]$ and $0 \neq \beta \in [n : M]$. Since M is a faithful, R-module then α and β are the vertices $\Gamma(R)$. Therefore $\alpha\beta = 0$, and so $[m : M][n : M]M = 0$. Hence $\Gamma(M)$ is a complete graph. □

The following example shows that the multiplication condition in the above theorem is not superfluous.

Example 3.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}_6$. So by El-Bast and Smith (Corollary 2.3, [11]), M is not a multiplication R-module. Also $\Gamma(M)$ is a complete graph, but $V(\Gamma(R)) = \emptyset$.



Corollary 3.3. *Let M be a faithful multiplication R-module with $|M| \neq 4$. If $\Gamma(R)$ is a complete graph, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Nil(M) = Nil(R)M = Z(R)M = T(M)$.*

Proof. Let $\Gamma(M)$ be a faithful multiplication R-module. By Theorem 3.1, $\Gamma(M)$ is a complete graph, and by Ghalandarzadeh and Malakooti (Theorem 2.11, [13]), $Nil(M) = T(M)$. Let $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, by Anderson and Livingston (Theorem 2.8, [1]), $Nil(R) = Z(R)$. Hence $Z(R)$ is an ideal of R and $T(M) = Z(R)M$. Therefore, we have that $Nil(M) = Nil(R)M = Z(R)M = T(M)$. □

Corollary 3.4. *Let M be a faithful multiplication R-module with $|M| \neq 4$. If $\Gamma(R)$ is a complete graph, then $|Min(M)| = 1$.*

Proof. Let M be a faithful multiplication R-module. By Theorem 3.1, $\Gamma(M)$ is a complete graph.Thus $T(M)$ is a submodule of M. We show that $\bigcup_{N \in Min(M)} N \subseteq T(M)$. Suppose that N be a prime submodule of M, such that $N \not\subseteq T(M)$. It will be sufficient to show that $N \notin Min(M)$. Since $N \not\subseteq T(M)$ there exists an element $x \in N$ such that $x \notin T(M)$. So there are $\alpha \in [x : M]$ and $m \in M$ such that $\alpha m \notin T(M)$. Now by puting $\Omega = \{\alpha^i \beta | i \geq 0$ and $\beta \in R \setminus [N : M]\}$, and similar to the proof of Proposition 2.6, one can check that $\bigcup_{N \in Min(M)} N \subseteq T(M)$. By Ghalandarzadeh and Malakooti (Theorem 2.11, [13]) and Theorem 2.4, we have

$$\bigcup_{N \in Min(M)} N \subseteq T(M) = Nil(M) = \bigcap_{N \in Min(M)} N,$$

which completes the proof. □

Theorem 3.5. *Let R be a Bézout ring and M be a multiplication R module such that $|M| \neq 4$ and $M \neq T(M)$; then $diam(\Gamma(M)) = diam(\Gamma(R))$.*

Proof. Let R be a Bézout ring and M be a multiplication R-module. By Theorem 3.1, $diam(\Gamma(M)) = 1$ if and only if $diam(\Gamma(R)) = 1$. Suppose that $diam(\Gamma(R)) = 2$ and $x, y \in T(M)^*$ such that $d(x, y) \neq 1$. Let $x = \sum_{i=1}^{n} \alpha_i m_i$ and $y = \sum_{j=1}^{m} \beta_j m_j$, where $0 \neq \alpha_i \in [x : M], 0 \neq \beta_j \in [y : M]$. Since R is a Bézout ring, $\sum_{i=1}^{n} R\alpha_i = R\alpha$ and $\sum_{j=1}^{m} R\beta_j = R\beta$, for some $\alpha, \beta \in R$. Hence there exist $m, m_0 \in M$ such that $x = \alpha m$, $y = \beta m_0$. Thus $\alpha, \beta \in Z(R)$. If $d(\alpha, \beta) = 1$, then $d(x, y) = 1$, and so we have a contradiction. Thus $d(\alpha, \beta) = 2$, so there exists $\gamma \in Z(R)^*$ such that $\alpha - \gamma - \beta$ is a path of length 2. Since $M \neq T(M)$, then there is $n \in M$ such that $\gamma n \in T(M)^*$. Therefore $\alpha m = x - \gamma n - y = \beta m$. is a path of length 2. So $d(x, y) = 2$ and $diam(\Gamma(M)) = 2$.

Suppose that $diam(\Gamma(M)) = 2$ and $\alpha, \beta \in Z(R)$ such that $d(\alpha, \beta) \neq 1$. So $\alpha\beta \neq 0$; since $M \neq T(M)$, then there is $n \in M$ such that $\alpha\beta n \neq 0$. Hence $\beta n \neq \alpha n \in T(M)^*$. If $d(\alpha n, \beta n) = 1$, then $[\alpha n : M]\beta n = 0$. So $\alpha\beta n = 0$, which is a contradiction. So $d(\alpha n, \beta n) = 2$, and there is $z = \gamma n \in T(M)^*$ such that $\alpha n - \gamma n - \beta n$, is a path of length 2. Thus $[\alpha n : M]\gamma n = 0 = [\beta n : M]\gamma n$, so $\alpha\gamma = 0 = \beta\gamma$ and $\alpha - \gamma - \beta$ is a path of length 2. Therefore $diam(\Gamma(R)) = 2$.

Now, let $diam(\Gamma(R)) = 3$, so $diam(\Gamma(M)) \geq 3$, and by Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $diam(\Gamma(M)) \leq 3$. Therefore $diam(\Gamma(M)) = 3$. If $diam(\Gamma(M)) = 3$, then $diam(\Gamma(R)) \geq 3$, and by Anderson and Livingston, (Theorem 2.3, [1]), $diam(\Gamma(R)) \leq 3$. Therefore $diam(\Gamma(R)) = 3$. Consequently $diam(\Gamma(M)) = diam(\Gamma(R))$. □

Lemma 3.6. *Let M be a reduced multiplication R-module and H be a finitely generated submodule of M. Then $Ann(H)M \neq 0$ if and only if $H \subseteq N$ for some $N \in Min(M)$.*

Proof. Let $Ann(H)M \neq 0$, so $Ann(H)M \nsubseteq Nil(M) = \bigcap_{N \in Min(M)} N$. Thus there exists $N_0 \in Min(M)$ such that $Ann(H)M \nsubseteq N_0$. Assume that $r \in R$ and $m \in M$ and $rm \in Ann(H)M$, but $rm \notin N_0$. Therefore $rm[H : M] = 0 \subseteq N_0$. Since $rm \notin N_0$, we have $H \subseteq N_0$.

To establish the reverse, let $N = PM \in Min(M)$, where $P = [N : M]$, and $H \subseteq N$. Since M is a reduced R-module, M_P will be a reduced R_P-module. We show that M_P has exactly one maximal submodule. Let M_P has two maximal submodules $S^{-1}H_1$ and $S^{-1}H_2$; so there exist two ideals $S^{-1}h_1$ and $S^{-1}h_2$ of $Max(S^{-1}R)$, such that $S^{-1}H_1 = S^{-1}h_1S^{-1}M$ and $S^{-1}H_2 = S^{-1}h_2S^{-1}M$. Since R_P is a local ring, $S^{-1}H_1 = S^{-1}H_2$. We know that $S^{-1}N$ is a proper submodule of $S^{-1}M$, and so by Theorem 2.1, $S^{-1}PS^{-1}M = S^{-1}N$ is the unique maximal submodule of M_P. Also, if $S^{-1}H_0$ is a prime submodule of M_P, then by Theorem 2.1, $S^{-1}H_0 \subseteq S^{-1}N$. By a routine argument $H_0 \subseteq N$, so $H_0 = N$; hence $S^{-1}H_0 = S^{-1}N$. Therefore by Theorem 2.4, $Nil(M_P) = S^{-1}N$. Since M_P is reduced, $Nil(M_P) = 0$. Thus $S^{-1}N = 0$. On the other hand, $H \subseteq N$; hence $S^{-1}H = 0$. Suppose that $H = \Sigma_{i=1}^{n} Rh_i$; so $\frac{h_i}{1} = 0$ for all $1 \leq i \leq n$. Hence there exists $s_i \in R \setminus P$ such that $s_ih_i = 0$. Let $s = s_1s_2 \cdots s_n$, thus $sH = 0$. If $sM = 0$ then $s \in [N : M] = P$, which is a contradiction. So there is an element $m \in M$ such that $0 \neq sm \in Ann(H)M$. $\square$

Theorem 2.6 in [15] characterizes the diameter of $\Gamma(R)$ in terms of the ideals of R. Our results obtained in Theorems 3.7 and 3.8 specifies the diameter of $\Gamma(M)$ in terms of minimal prime submodules of a multiplication module M over a von Neumann regular ring.

Theorem 3.7. *Let R be a von Neumann regular ring and M be a multiplication R-module. If M has more than two minimal prime submodules and $T(M)$ is not a submodule of M, then $diam(\Gamma(M)) = 3$.*

Proof. Let m, n be two distinct elements of $T(M)^*$ and $Ann(Rm + Rn) = 0$. Hence M is faithful. First, suppose that $[m : M][n : M]M \neq 0$, so $d(m, n) \neq 1$. If $d(m, n) = 2$, then there exists a vertex $x \in T(M)^*$ such that $m - x - n$ is a path. Thus

$$[m : M][x : M]M = 0 = [x : M][n : M]M.$$

Accordingly $[x : M](Rm + Rn) = 0$, and so $[x : M] \subseteq Ann(Rm + Rn) = 0$. Which is a contradiction. We shall now assume that $d(m, n) \neq 2$. By Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\Gamma(M)$ is connected with $diam(\Gamma(M)) \leq 3$; therefore $d(m, n) = 3$. Next, assume $[m : M][n : M]M = 0$, then by Proposition 2.6, $m, n \in \bigcup_{N \in Min(M)} N$. Since $Ann(Rm+Rn)M = 0$, by Lemma 3.6, m and n belong to two distinct minimal prime submodules. Suppose that P, N and Q are distinct minimal prime submodules of M such that $m \in P \setminus (Q \cup N)$ and $n \in (Q \cap N) \setminus P$. Hence $[m : M]M \nsubseteq N$; thus $\alpha m \notin N$ for some $\alpha \in [m : M]$ and $m \in M$. Let $x \in (Q \cap P) \setminus N$. A simple check yields that $\alpha^2 x \neq 0$. On the other hand, since $[m : M][n : M]M = 0$, we have $\alpha(n + \alpha x) = \alpha^2 x$. Therefore $0 \neq \alpha^2 x \in [m : M][n + \alpha x : M]M$. Also, by a routine argument, we have $Rm + Rn = Rm + R(n + \alpha x)$. So $Ann(Rm + R(n + \alpha x)) = 0$. Similar to the above argument, we have $d(m, (n + \alpha x)) = 3$. Consequently $diam(\Gamma(M)) = 3$. $\square$

Theorem 3.8. *Let R be a von Neumann regular ring and M be a multiplication R-module. If $T(M)$ is not a submodule of M, then $diam(\Gamma(M)) \leq 2$ if and only if M has exactly two minimal prime submodules.*

Proof. Suppose that $diam(\Gamma(M)) \leq 2$, and $T(M)$ is not a submodule of M, so there exist $m, n \in T(M)^*$ with $Ann(Rm + Rn) = 0$. So M is faithful and by Ghalandarzadeh and Malakooti (Theorem 2.6, [13]), $\Gamma(M)$ is connected. Now since $\Gamma(M)$ is a connected graph and $T(M)$ is not a submodule of M, by Proposition 2.6 and Lemma 3.6, there are at least two distinct minimal prime submodules P and Q of M such that $m \in P \setminus Q$ and $n \in Q \setminus P$. On the other hand, by Theorem 3.7, M can not have more than two minimal prime submodules; therefore M has exactly two minimal prime submodules. Conversely, suppose that P and Q be only two minimal prime submodules of M. By Proposition 2.6, $T(M) = P \cup Q$. Assume that $m, n \in T(M)^*$ such that $m \in P \setminus Q$ and $n \in Q \setminus P$. Thus $[m : M][n : M]M \subseteq P \cap Q = Nil(M) = 0$, by Lemma 2.5. So $d(m, n) = 1$. Also if $m, n \in P$, then $Rm + Rn \subseteq P$. By Lemma 3.6, $Ann(Rm + Rn)M \neq 0$; therefore there is $0 \neq \alpha \in R$ such that $\alpha m = \alpha n = 0$. On the other hand, there exists a non-zero element x of M such that $\alpha x \neq 0$ and so $m - \alpha x - n$ is a path; hence $d(m, n) = 2$, thus $diam(\Gamma(M)) \leq 2$. Moreover, if $n, m \in Q$, then similarly $diam(\Gamma(M)) \leq 2$. $\square$

As an immediate consequence from Theorem 3.5 and Theorem 3.8, we obtain the following result.

Corollary 3.9. *Let R be a von Neumann regular ring and let M be a multiplication R-module. If $T(M)$ is not a submodule of M, then M has exactly two minimal prime submodules if and only if R has exactly two minimal prime ideals.*

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