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On classifications of transformation semigroups: Indicator sequences and indicator topological spaces

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Abstract. In this paper considering a transformation semigroup with finite height we define the notion of indicator sequence in such a way that any two transformation semigroups with the same indicator sequence have the same height. Also related to any transformation semigroup a topological space, called indicator topological space, is defined in such a way that transformation semigroups with homeomorphic indicator topological spaces have the same height. Moreover any two transformation semigroups with homeomorphic indicator topological spaces and finite height have the same indicator sequences.

1. Introduction

Classification of special subclasses of topological transformation groups is an old idea (see [6] and [7]), e.g. in [7] at the first sentence of the Introduction we read "*The classification of minimal sets is one of the goals of topological dynamics.*"

In this paper we want to introduce some tools for classifying the class of all transformation semigroups using some known tools. Height of a transformation semigroup (X, S), $\mathrm{h}(X, S)$, has been introduced firstly in [8]. In this paper using the concept of height of a transformation semigroup where $\mathrm{h}(X, S)$ is finite we correspond a finite sequence $(n_0, \ldots, n_{\mathrm{h}(X,S)})$, indicator sequence of (X, S), to the transformation semigroup (X, S), for instance (X, S) is a finite disjoint union of minimal sets if and only if its indicator sequence is $(0, \ldots, 0)$. So the concept of indicator sequence is a tool to make some partitions in the class of all transformation semigroups with the same finite height. For the next step we create another fragmentation, i.e. in the class of all transformation semigroups with the same indicator sequence $(n_0, \ldots, n_m)$ we correspond an $m + 1-$element T_0 topological space, indicator topology of (X, S), to the transformation semigroup (X, S). We see if (X, S) and (Y, T) have homeomorphic finite indicator topologies, then they have the same indicator sequence. Also there are transformation semigroups with the same indicator sequence and non-homeomorphic indicator topological spaces.

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2. Preliminaries

By a *(right) transformation semigroup* we mean a triple (X, S, θ) or simply (X, S) where X is a topological space (called phase space), S is a topological semigroup (called phase semigroup) with identity e and $\theta : X \times S \to X$ ($\theta(x, s) = xs$) is a continuous map such that for each $x \in X$ and each $s, t \in S$ we have $xe = x$ and $x(st) = (xs)t$. *In the following, all phase spaces are considered compact Hausdorff and all phase semigroups are considered discrete, unless it is mentioned.*
In the transformation semigroup (X, S) a nonempty subset Z of X is called *invariant* if $ZS \subseteq Z$. A closed invariant subset Z of X is called *minimal* if it does not have any proper subset which is a closed invariant subset of X. Let $\mathcal{R}$ be a closed invariant equivalence relation on X, then $X/\mathcal{R}$ is a compact Hausdorff topological space and $(X/\mathcal{R}, S)$ is a transformation semigroup, where $[x]_{\mathcal{R}}s := [xs]_{\mathcal{R}}$ ($[x]_{\mathcal{R}}$ is the equivalence class of x under $\mathcal{R}$). For a closed invariant subset C of X, $(C \times C) \cup \Delta_X$ is a closed invariant equivalence relation on X, we will denote $(X/((C \times C) \cup \Delta_X), S)$ by $(X/C, S)$.
A continuous map $\varphi : (X, S) \to (Y, S)$ is called a *homomorphism* if $\varphi(xs) = \varphi(x)s$ ($x \in X, s \in S$). If there exists an onto homomorphism $\varphi : (X, S) \to (Y, S)$, then (X, S) is called an *extension* of (Y, S) and (Y, S) is called a *factor* of (X, S). A one-to-one and onto homomorphism $\varphi : (X, S) \to (Y, S)$ is called an *isomorphism*. (X, S), (Y, S) are called *isomorphic* transformation semigroups if there exists an isomorphism $\varphi : (X, S) \to (Y, S)$. If $\varphi : (X, S) \to (Y, S)$ is an onto homomorphism, then $R(\varphi) := \{(x, y) \in X \times X : \varphi(x) = \varphi(y)\}$ is a closed invariant equivalence relation on X, moreover $(X/R(\varphi), S)$ and (Y, S) are isomorphic transformation semigroups.
If $\mathcal{D}$ is a subclass of all transformation semigroups with phase semigroup S, the transformation semigroup (Z, S) is called a *universal* transformation semigroup for class $\mathcal{D}$ if all of the elements of $\mathcal{D}$ are factors of (Z, S). Note that by this definition there is no need for $(Z, S) \in \mathcal{D}$.
For more details on transformation semigroups, we suggest [5] and [9].
Let Z be a closed invariant subset of the transformation semigroup (X, S), define height of Z by $\mathrm{h}(Z, S) := \sup\{n \geq 0$: there exists a chain like $Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z$ of closed invariant subsets of X and $Z_i \neq Z_{i+1} (i < n)\}$ ([8], Definition 1]), it is known that $\mathrm{h}(Z, S) = \sup\{n \geq 0$: there exist $z_0, \ldots, z_n \in Z$ with $z_i \notin \bigcup_{j<i} \overline{z_j S}$ for all $i \leq n\}$ ([8], Lemma 8). Moreover $\mathrm{h}(Z, S) = 0$ if and only if Z is a minimal subset of X. In addition $\mathrm{h}(Z, S) = 1$ if and only if one of the following conditions hold ([8], Theorem 9):

- Z is disjoint union of two minimal subsets;
- Z contains a unique minimal proper subset W and for any $x \in Z - W$ we have $\overline{xS} = Z$.

Moreover if C is a closed invariant subset of X, then $\mathrm{h}(C, S) \leq \mathrm{h}(X, S)$ (monotonicity of h), and if $\mathrm{h}(C, S) = \mathrm{h}(X, S) < \infty$, then $C = X$ ([8], Note 4).

Convention 2.1. *In the following text by "$\subset$" we mean strict inclusion.*

3. More properties of h(*X*, *S*)

In this section we consider some properties of the height of a transformation semigroup which are necessary for the other sections.

Lemma 3.1. *In the transformation semigroup (X, S), if $Z_0 \subset \cdots \subset Z_n$ ($n \in \mathbb{N} \cup \{0\}$) is a maximal chain of closed invariant subsets of X, then for every $x, y \in Z_{i+1} - Z_i$ ($i < n$) we have $Z_{i+1} - Z_i \subseteq \overline{xS} = \overline{yS}$.*

Proof. Suppose $x \in Z_{i+1} - Z_i$ thus $Z_i \subset Z_i \cup \overline{xS} \subseteq Z_{i+1}$, since $Z_0 \subset \cdots \subset Z_n$ is a maximal chain, $Z_i \cup \overline{xS} = Z_{i+1}$ and $Z_{i+1} - Z_i \subseteq \overline{xS}$, therefore for any $y \in Z_{i+1} - Z_i$ we have $y \in \overline{xS}$ and $\overline{yS} \subseteq \overline{xS}$. The similar method shows $\overline{xS} \subseteq \overline{yS}$, so $\overline{yS} = \overline{xS}$. □

The following theorem has a key role in the rest theorems and other sections, indeed this is the main theorem of this section.

Theorem 3.2. *In the transformation semigroup* (X, S) *for* $n \in \mathbb{N} \cup \{0\}$ *the following statements are equivalent:*

(1) *there exists a maximal chain* $Z_0 \subset \cdots \subset Z_n$ *of closed invariant subsets of* X*;*
(2) $\{\overline{xS} : x \in X\}$ *has exactly* $n+1$ *elements;*
(3) *for any maximal chain of closed invariant subsets of* X *like* $\mathcal{M}$, $\mathcal{M}$ *has exactly* $n+1$ *elements;*
(4) $\mathrm{h}(X, S) = n$.

Proof. (1) ⇒ (2): Suppose $Z_0 \subset \cdots \subset Z_n$ is a maximal chain of closed invariant subsets of X, and $z_0 \in Z_0$, $z_1 \in Z_1 - Z_0$, ..., $z_n \in Z_n - Z_{n-1}$, then by Lemma 3.1, and minimality of Z_0 (thus $\overline{z_0 S} = Z_0$) we have $\{\overline{xS} : x \in X\} = \{\overline{z_i S} : 0 \leq i \leq n\}$, On the other hand, if $i < j$, then $\overline{z_i S} \subseteq Z_i$ and $z_j \notin Z_i$, thus $\overline{z_i S} \neq \overline{z_j S}$, which shows that $\{\overline{z_i S} : 0 \leq i \leq n\}$ has exactly $n+1$ elements.
(2) ⇒ (3): Suppose $\{\overline{xS} : x \in X\} = \{\overline{x_0 S}, \ldots, \overline{x_n S}\}$ has exactly $n+1$ elements and $\mathcal{M}$ is a maximal chain of closed invariant subsets of X. Since for any closed invariant subset of X, like Z we have $Z = \bigcup\{\overline{zS} : z \in Z\}$, any element of $\mathcal{M}$ is a union of a subset of $\{\overline{x_0 S}, \ldots, \overline{x_n S}\}$, so $\mathcal{M}$ is finite. Using the proof of (1) ⇒ (2) leads us to the desired result.
(3) ⇒ (4), (4) ⇒ (1): It is clear by the definition of $\mathrm{h}(X, S)$ and the fact that by Zorn's Lemma any chain of closed invariant subsets of X is included in a maximal chain of closed invariant subsets of X. □

Let (X, S), (Y, T) be transformation semigroups, then $(X \times Y, S \times Y)$ is a transformation semigroup where $(x, y)(s, t) := (xs, yt)$ $((x, y) \in X \times Y, (s, t) \in S \times T)$, in the following theorem we deal with $\mathrm{h}(X \times Y, S \times T)$.

Theorem 3.3. *In the transformation semigroups* (X, S), (Y, T) *we have*

$$\mathrm{h}(X \times Y, S \times T) + 1 = (\mathrm{h}(X, S) + 1)(\mathrm{h}(Y, T) + 1).$$

Proof. Suppose $n = \mathrm{h}(X, S) < \infty$ and $m = \mathrm{h}(Y, T) < \infty$, thus $\{\overline{xS} : x \in X\}$ has exactly $n+1$ elements and $\{\overline{yT} : y \in Y\}$ has exactly $m+1$ elements (use Theorem 3.2), therefore $\{\overline{(x, y)(S \times T)} : (x, y) \in X \times Y\} = \{\overline{xS} \times \overline{yT} : x \in X, y \in Y\}$ has exactly $(n+1)(m+1)$ elements, which shows $\mathrm{h}(X \times Y, S \times T) + 1 = (n+1)(m+1) = (\mathrm{h}(X, S) + 1)(\mathrm{h}(Y, T) + 1)$. □

Note 3.4. If $\{(X_\alpha, S_\alpha) : \alpha \in \Gamma\}$ is a nonempty collection of transformation semigroups, then $(\prod_{\alpha \in \Gamma} X_\alpha, \prod_{\alpha \in \Gamma} S_\alpha)$ is a transformation semigroup where:

$$((x_\alpha)_{\alpha \in \Gamma}(s_\alpha)_{\alpha \in \Gamma}) := (x_\alpha s_\alpha)_{\alpha \in \Gamma} \quad ((x_\alpha)_{\alpha \in \Gamma} \in \prod_{\alpha \in \Gamma} X_\alpha, (s_\alpha)_{\alpha \in \Gamma} \in \prod_{\alpha \in \Gamma} S_\alpha);$$

moreover we know $\overline{(x_\alpha)_{\alpha \in \Gamma} \prod_{\alpha \in \Gamma} S_\alpha} = \prod_{\alpha \in \Gamma} \overline{x_\alpha S_\alpha}$; thus $(\prod_{\alpha \in \Gamma} X_\alpha, \prod_{\alpha \in \Gamma} S_\alpha)$ is minimal if and only if for all $\alpha \in \Gamma$, (X_α, S_α) is minimal. Using a similar method described in Theorem 3.3 the following statements are equivalent:

1. $\mathrm{h}(\prod_{\alpha \in \Gamma} X_\alpha, \prod_{\alpha \in \Gamma} S_\alpha) < \infty$;
2. there exist $\alpha_1, \ldots, \alpha_n \in \Gamma$ such that $\max\{\mathrm{h}(X_{\alpha_i}, S_{\alpha_i}) : 1 \leq i \leq n\} < \infty$ and for all $\alpha \in \Gamma - \{\alpha_1, \ldots, \alpha_n\}$, (X_α, S_α) is minimal (i.e. $\mathrm{h}(X_\alpha, S_\alpha) = 0$).

And if (2) holds then $\mathrm{h}(\prod_{\alpha \in \Gamma} X_\alpha, \prod_{\alpha \in \Gamma} S_\alpha) = \mathrm{h}(\prod_{i=1}^{n} X_{\alpha_i}, \prod_{i=1}^{n} S_{\alpha_i})$.

Remark 3.5. In the transformation semigroup (X, S), let C be a closed invariant subset of X. There is a monotone (under inclusion relation) one-to-one correspondence between the set all closed invariant subsets of X containing C, and the set of all closed invariant subsets of X/C containing the singleton C/C.

Now we are ready to give attention to quotients, i.e. in the transformation semigroup (X, S) we want to consider $\mathrm{h}(X/C, S)$ when C is a closed invariant subset of X.

Theorem 3.6. *In the transformation semigroup (X, S), let C be a closed invariant subset of X, then* $\mathrm{h}(X/C, S) + \mathrm{h}(C, S) = \mathrm{h}(X, S)$.

Proof. Suppose $\mathrm{h}(C, S) < \infty$ (if $\mathrm{h}(C, S) = \infty$, then $\mathrm{h}(X, S) = \infty$ too by definition, and we are done) and $C_0 \subset C_1 \subset \cdots \subset C_n = C$ is a maximal chain of closed invariant subsets of C, this chain can be extended to a maximal chain of closed invariant subsets of X, like $\mathcal{M}$. $\{Z/C : Z \in \mathcal{M}, Z \neq C_0, \ldots, C_{n-1}\}$ is a maximal chain of closed invariant subsets of X/C, so if $\mathrm{h}(X, S) = \infty$, then $\mathrm{h}(X/C, S) = \infty$ too, and $\mathrm{h}(X/C, S)+\mathrm{h}(C, S) = \mathrm{h}(X, S)$. Now suppose $\mathrm{h}(X, S) < \infty$, using Theorem 3.2 we have $\mathrm{h}(X, S) = |\mathcal{M}| - 1 = |\{C_0, \ldots, C_{n-1}\}| + |\{Z \in \mathcal{M} : Z \neq C_0, \ldots, C_{n-1}\}| - 1 = |\{C_0, \ldots, C_{n-1}\}| + |\{Z/C : Z \in \mathcal{M}, Z \neq C_0, \ldots, C_{n-1}\}| - 1 = \mathrm{h}(C, S) + \mathrm{h}(X/C, S)$. □

Using the above theorem we find some other equivalent statements to the finiteness of the height of a transformation semigroup in terms of quotient spaces.

Theorem 3.7. *In the transformation semigroup (X, S), let $C_0 \subset \cdots \subset C_n = X$ be a chain of closed invariant subsets of X, the following statements are equivalent:*

1. *$C_0 \subset \cdots \subset C_n$ is a maximal chain of closed invariant subsets of X;*
2. *C_0 is minimal and for any $x \in C_{i+1} - C_i$ we have $C_{i+1} - C_i \subseteq \overline{xS}$ $(i < n)$;*
3. *if $x_0, \ldots, x_n \in X$ are such that $x_0 \in C_0$ and $x_{i+1} \in C_{i+1} - C_i$ $(i < n)$, then $\{\overline{xS} : x \in X\} = \{\overline{x_iS} : 0 \leq i \leq n\}$;*
4. *$\mathrm{h}(X, S) = n$;*
5. *$\mathrm{h}(C_0, S) = 0$ and $\mathrm{h}(C_{i+1}/C_i, S) = 1$ for all $i < n$.*

Proof. (1) ⇒ (2): Use Lemma 3.1.
(2) ⇒ (3): If $x \in C_0$ then $\overline{xS} = \overline{x_0S}$ and if $x \in X - C_0$ then there exists $i < n$ such that $x \in C_{i+1} - C_i$ and then $\overline{xS} = \overline{x_{i+1}S}$.
(3) ⇒ (4): Use Theorem 3.2.
(4) ⇒ (5): Since $\mathrm{h}(C_0, S) < \mathrm{h}(C_1, S) < \cdots < \mathrm{h}(C_n, S) = n$, $\mathrm{h}(C_i, S) = i$ for all $0 \leq i \leq n$. Using Theorem 3.6 we are done.
(5) ⇒ (1): By Theorem 3.6 and induction we have $\mathrm{h}(C_i, S) = i$ for any $0 \leq i \leq n$, in particular $\mathrm{h}(X, S) = n$ and then $C_0 \subset \cdots \subset C_n$ is a maximal chain of closed invariant subsets of X. □

4. Indicator sequence of a transformation semigroup (X, S) with $\mathrm{h}(X, S) < \infty$

In this section corresponding to a transformation semigroup (X, S) with $\mathrm{h}(X, S) < \infty$, we define a sequence $(n_0, \ldots, n_{\mathrm{h}(X,S)})$ of nonnegative integers. In this way transformation semigroups with the same indicator sequence have the same height, but the converse is not true in general.

Definition 4.1. Let (X, S) be a transformation semigroup with $\mathrm{h}(X, S) = n < \infty$. By Theorem 3.2, there exist $x_0, \ldots, x_n \in X$ with $\{\overline{xS} : x \in X\} = \{\overline{x_iS} : 0 \leq i \leq n\}$ (thus $\overline{x_iS}$s are distinct). Without loss of generality just by a rearrangement we may suppose $\mathrm{h}(\overline{x_0S}, S) \leq \cdots \leq \mathrm{h}(\overline{x_nS}, S)$. We call $(\mathrm{h}(\overline{x_0S}, S), \ldots, \mathrm{h}(\overline{x_nS}, S))$ the *indicator sequence of* (X, S) and denote it by $\mathrm{seq}(X, S)$.

In the following lemma we find a necessary condition for a finite sequence of integers to be the indicator sequence of a transformation semigroup. In fact, we see that this condition is sufficient too.

Lemma 4.2. *In the transformation semigroup (X, S) with $m = \mathrm{h}(X, S) < \infty$ and $\mathrm{seq}(X, S) = (n_0, \ldots, n_m)$, we have $n_i \leq i$ for all $i \leq m$.*

Proof. Let $x_0, \ldots, x_m \in X$ be the ones in Definition 4.1. Hence $\mathrm{h}(\overline{x_iS}, S) = n_i$ and $n_0 \leq n_1 \leq \cdots \leq n_m$. Thus if $i < j$, then $\mathrm{h}(\overline{x_iS}, S) \leq \mathrm{h}(\overline{x_jS}, S)$. We claim $x_j \notin \overline{x_iS}$, otherwise $\overline{x_jS} \subseteq \overline{x_iS}$ which implies $\mathrm{h}(\overline{x_jS}, S) \leq \mathrm{h}(\overline{x_iS}, S)$ and $\mathrm{h}(\overline{x_jS}, S) = \mathrm{h}(\overline{x_iS}, S)$. Thus $\overline{x_iS} = \overline{x_jS}$, which is a contradiction. Therefore $\overline{x_0S} \subset \bigcup\{\overline{x_jS} : j = 0, 1\} \subset \cdots \subset \bigcup\{\overline{x_jS} : 0 \leq j \leq m\}$ is a chain of $m + 1$ closed invariant subsets of X. Since $\mathrm{h}(X, S) = m$, by Theorem 3.7 this chain is a maximal chain. So for any $i \in \{0, \ldots, m\}$, $n_i = \mathrm{h}(\overline{x_iS}, S) \leq \mathrm{h}(\bigcup\{\overline{x_jS} : 0 \leq j \leq i\}, S) = i$. □

Lemma 4.3. *Let $(n_0, \dots, n_m)$ be a sequence of nonnegative integers such that $n_0 \le n_1 \le \cdots \le n_m$ and $0 \le n_i \le i$ for all $i \le m$. Then for $X_m = \{0, \dots, m\}$ with discrete topology there exists a semigroup $T_m \subseteq X_m^{X_m}$ (with discrete topology and under composition operation) such that the following statements hold:*

(1) $iT_m \subseteq \{0, \dots, i\}$, $\mathrm{h}(\overline{iT_m}, T_m) = n_i$ and $|iT_m| = n_i + 1$, for all $i \le m$;

(2) $\{0\} \subset \{0,1\} \subset \cdots \subset \{0, \dots, m\}$ is a maximal chain of closed invariant subsets of X_m, and $\mathrm{seq}(X_m, T_m) = (n_0, \dots, n_m)$.

Proof. We prove by induction on m. For $m = 0$ we have $n_0 = 0$ and then $T_0 = \{\mathrm{id}_{\{0\}}\}$ satisfies conditions (1) and (2). Suppose statements (1) and (2) are true for m and $(n_0, \dots, n_{m+1})$ is a sequence of nonnegative integers such that $n_0 \le n_1 \le \cdots \le n_{m+1}$ and $0 \le n_i \le i$ for all $i \le m+1$. Using induction hypothesis there exists $T_m \subseteq {X_m}^{X_m}$ satisfying conditions (1) and (2). Let $T_{m+1} := \{\zeta \in X_{m+1}^{X_{m+1}} : \zeta|_{X_m} \in T_m$ and $((m+1)\zeta < n_{m+1}$ or $(m+1)\zeta = m+1)\}$. Thus transformation semigroup (X_{m+1}, T_{m+1}) satisfies conditions (1) and (2) for $m+1$ instead of m. □

The following theorem is the main theorem of this section in which all possible indicator sequences are introduced.

Theorem 4.4. *Let $(n_0, \dots, n_m)$ be a sequence of nonnegative integers. There exists a transformation semigroup (X, S) with $\mathrm{seq}(X, S) = (n_0, \dots, n_m)$ if and only if $n_0 \le n_1 \le \cdots \le n_m$ and $0 \le n_i \le i$ for all $i \le m$.*

Proof. Use Lemma 4.2 and Lemma 4.3. □

The following theorem shows some interactions between the concept of indicator sequence and subspaces or finite products of transformation semigroups.

Theorem 4.5. *In the transformation semigroups (X, S) and (Y, T) suppose $\mathrm{h}(X, S) < \infty$, $\mathrm{h}(Y, T) < \infty$, $\mathrm{seq}(X, S) = (q_0, \dots, q_n)$, $\mathrm{seq}(Y, T) = (r_0, \dots, r_m)$, and Z is a closed invariant subset of X. Then:*

1. $\mathrm{seq}(Z, S)$ *is a subsequence of* $(q_0, \dots, q_n)$*;*
2. $\mathrm{seq}(X \times Y, S \times T)$ *is a rearrangement of* $((q_i + 1)(r_j + 1) - 1)_{0 \le i \le n, 0 \le j \le m}$.

Proof. Using Theorem 3.2 and that $\{\overline{zS} : z \in Z\} \subseteq \{\overline{xS} : x \in X\}$, (1) holds. On the other hand $\{\overline{(x,y)(S \times T)} : (x, y) \in X \times Y\} = \{\overline{xS} \times \overline{yT} : x \in X, y \in Y\}$ and by Theorem 3.3 for any $(x, y) \in X \times Y$, $\mathrm{h}(\overline{(x,y)(S \times T)}, S \times T) + 1 = (\mathrm{h}(\overline{xS}, S) + 1)(\mathrm{h}(\overline{yT}, T) + 1)$. Thus (2) holds. □

Remark 4.6. Define the sequence of Catalan numbers, $\{C_n\}_{n \ge 0}$, by (see e.g. [1], Section 2.6):

$$C_n = (n+1)^{-1} \binom{2n}{n} \quad (n \ge 0);$$

or define equivalently in a recursive way by:

$$C_0 = C_1 = 1, C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} \quad (n \ge 2).$$

In the following theorem we find the exact number of all indicator sequences $(n_0, \dots, n_m)$, for $m \in \mathbb{N} \cup \{0\}$.

Theorem 4.7. *Let $m \in \mathbb{N} \cup \{0\}$, then $|\{(n_0, \dots, n_m) : \exists (X, S)\ \mathrm{h}(X, S) = m\}| = (m+2)^{-1} \binom{2m+2}{m+1}$ (i.e. C_{m+1}).*

Proof. For $m \in \mathbb{N} \cup \{0\}$ set $B_m = \{(n_0, \dots, n_m) : \exists (X, S)\ \mathrm{h}(X, S) = m\}$, by Theorem 4.4, $B_m = \{(n_0, \dots, n_m) : 0 \leq n_0 \leq n_1 \leq \cdots \leq n_m$ and $n_i \leq i$ for all $i \in \{0, \dots, m\}\}$. Set $\beta_m := |B_m|$. It is easy to see $\beta_0 = 1$ and $\beta_1 = 2$, moreover set $\beta_{-1} := 1$. We claim:

$$\beta_m = \sum_{k=0}^{m} \beta_{k-1}\beta_{m-k-1} \qquad (*)$$

Obviously $(*)$ is true for $m = 0, 1$, so suppose $m \geq 2$. For $k \in \{0, \dots, m\}$ let $B_m^k := \{(n_0, \dots, n_m) \in B_m : n_k = k$ and for all $i > k$, $n_i < i\}$. B_m is disjoint union of $B_m^0, \dots, B_m^m$. Moreover $|B_m^0| = |B_m^m| = \beta_{m-1}$ and $|B_m^k| = |B_k^k||B_{m-k}^0| = \beta_{k-1}\beta_{m-k-1}$ for $k = 1, \dots, m-1$. Thus

$$\beta_m = |B_m| = \sum_{k=0}^{m} |B_m^k| = \sum_{k=0}^{m} \beta_{k-1}\beta_{m-k-1}.$$

Now use induction on m. Clearly $\beta_0 = C_1$ and $\beta_1 = C_2$, suppose $\beta_i = C_{i+1}$ for all $i \in \{0, \dots, m\}$, thus:

$$\beta_{m+1} = \sum_{k=0}^{m+1} \beta_{k-1}\beta_{m-k} = \sum_{k=0}^{m+1} C_k C_{m+1-k} = C_{m+2},$$

and we are done. □

5. Indicator topological space of a transformation semigroup

In this section for every transformation semigroup (X, S) we introduce a topological space $\widehat{X}$, indeed $\widehat{X}$ is just $\{\overline{xS} : x \in X\}$ equipped with a T_0 Alexandroff topology which we name *indicator topology on* $\widehat{X}$ *corresponding to* (X, S). The end of this section is devoted to transformation semigroups with finite height.

Definition 5.1. In the transformation semigroup (X, S) define $\widehat{X} = \{\widehat{x} : x \in X\}$ where $\widehat{x} = \overline{xS}$. Equip $\widehat{X}$ with topology generated by topological basis $\{\{\widehat{y} : y \in \overline{xS}\} : x \in X\}$, we call this topological space *the indicator topological space of* (X, S).

Remark 5.2. A topological space is called Alexandroff if any nonempty intersection of open sets is open. In Alexandroff topological space A, for any $a \in A$, denote the smallest open neighborhood of a by $V_A(a)$. Thus in the transformation semigroup (X, S) for each $x \in X$, $V_{\widehat{X}}(\widehat{x}) = \{\widehat{y} : y \in \overline{xS}\}$.

The following theorem translates some ideas from transformation semigroups theory into general topology.

Theorem 5.3. *In the transformation semigroup* (X, S) *we have:*

1. $\widehat{X}$ *is a* T_0 *Alexandroff space;*
2. *suppose* $x \in X$*, then:*

$$\mathrm{h}(\overline{xS}, S) = \begin{cases} |V_{\widehat{X}}(\widehat{x})| - 1 & V_{\widehat{X}}(\widehat{x}) \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

3. *suppose* $x \in X$*,* $\widehat{x}$ *is an isolated point of* $\widehat{X}$ *if and only if* $\overline{xS}$ *is a minimal subset of* (X, S)*;*
4. *the following statements are equivalent:*
   - *(a)* X *is the union of its minimal sets (i.e. for each* $x \in X$*,* $\overline{xS}$ *is minimal),*
   - *(b)* $\widehat{X}$ *is* T_1*,*
   - *(c)* $\widehat{X}$ *is discrete;*

5. *X is point transitive if and only if there exists $x \in X$ with $\widehat{x} \in \overline{A}$ for any nonempty subset A of $\widehat{X}$;*
6. *if Z is a closed invariant subset of X, then $\widehat{Z}$ is a subspace of $\widehat{X}$.*

Proof.

1. Since for any $x \in X$, $\{\widehat{y} : y \in \overline{xS}\}$ is the smallest open set containing $\widehat{x}$ and if $\widehat{y} \in \widehat{X} - \{\widehat{x}\}$ then $x \notin \overline{yS}$ or $y \notin \overline{xS}$.
2. Use (2) in Theorem 3.2.
3. Use (2).
4. Use (3) and the fact that an Alexandroff space is T_1 if and only if it is discrete.
5. By definition X is point transitive if and only if there exists $x \in X$ such that $\overline{xS} = X$. So X is point transitive if and only if there exists $x \in X$ such that $V_{\widehat{X}}(\widehat{x}) = \widehat{X}$. Thus (5) holds.
6. Since for any $z \in Z$ we have $V_{\widehat{Z}}(\widehat{z}) = V_{\widehat{X}}(\widehat{z})$.

□

As we will see in the following statements the concept of indicator topology has a good relationship with some other facts in transformation semigroups. For instance, the indicator topology of the product space is the product of indicator topologies under box topology and the indicator topology of the quotient space is the quotient of indicator topologies.

Proposition 5.4. *Let $\{(X_\alpha, S_\alpha)\}_{\alpha\in\Gamma}$ be a nonempty collection of transformation semigroups. In the transformation semigroup $(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$, $\widehat{\prod_{\alpha\in\Gamma} X_\alpha} = \prod_{\alpha\in\Gamma} \widehat{X_\alpha}$ whenever $\prod_{\alpha\in\Gamma} \widehat{X_\alpha}$ is equipped with the box topology.*

Proof. Since $V_{\widehat{\prod_{\alpha\in\Gamma} X_\alpha}}(\widehat{(x_\alpha)_{\alpha\in\Gamma}}) = \prod_{\alpha\in\Gamma} V_{\widehat{X_\alpha}}(\widehat{x_\alpha})$ for any $(x_\alpha)_{\alpha\in\Gamma} \in \prod_{\alpha\in\Gamma} X_\alpha$. □

Corollary 5.5. *Let $\{(X_\alpha, S_\alpha)\}_{\alpha\in\Gamma}$ be a nonempty collection of transformation semigroups. In the transformation semigroup $(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} S_\alpha)$, $\widehat{\prod_{\alpha\in\Gamma} X_\alpha} = \prod_{\alpha\in\Gamma} \widehat{X_\alpha}$ whenever $\prod_{\alpha\in\Gamma} \widehat{X_\alpha}$ equipped with product topology if and only if $\{\alpha \in \Gamma : (X_\alpha, S_\alpha)$ is not minimal$\}$ is finite.*

Proof. Using item (1) in Theorem 5.3, for each $\alpha \in \Gamma$, $\widehat{X_\alpha}$ is a T_0 space, thus product and box topologies on $\prod_{\alpha\in\Gamma} \widehat{X_\alpha}$ coincide if and only if $\{\alpha \in \Gamma : \widehat{X_\alpha}$ is not a singleton$\}$ is finite. On the other hand by item (3) in Theorem 5.3, $\widehat{X_\alpha}$ is a singleton if and only if (X_α, S_α) is minimal, now use Proposition 5.4. □

Lemma 5.6. *In the transformation semigroups (X, S), (Y, T) let $\varphi : X \to Y$ be an onto map such that $\varphi(\overline{xS}) = \overline{\varphi(x)T}$ $(x \in X)$. Then $\widehat{\varphi} : \widehat{X} \to \widehat{Y}$ with $\widehat{\varphi}(\widehat{x}) = \widehat{\varphi(x)}$ $(x \in X)$ is an open quotient mapping.*

Proof. Since $\widehat{\varphi}$ is onto it is enough to show that $\widehat{\varphi}$ is open and continuous. For each $x \in X$ we have $\widehat{\varphi}(\widehat{x}) = \widehat{\varphi(x)} = \varphi(\widehat{x})$ so $\widehat{\varphi}(V_{\widehat{X}}(\widehat{x})) = V_{\widehat{Y}}(\widehat{\varphi}(\widehat{x}))$ and then $\widehat{\varphi}$ is open and continuous. □

Theorem 5.7. *In the transformation semigroups (X, S), (Y, S) let $\varphi : X \to Y$ be an onto homomorphism, then $\widehat{Y}$ is homeomophic to a quotient of $\widehat{X}$.*

Proof. Use Lemma 5.6. □

Note 5.8. Using the above theorem in the transformation semigroup (X, S) let $\mathcal{R}$ be a closed invariant equivalence relation on X, then $\widehat{X/\mathcal{R}}$ is homeomophic to a quotient of $\widehat{X}$(consider natural quotient mapping from (X, S) to $(X/\mathcal{R}, S)$, which is an onto homomorphism too). Therefore if C is a closed invariant subset of X, then $\widehat{X/C}$ is homeomophic to a quotient of $\widehat{X}$, indeed $\widehat{X/C}$ and $\widehat{X}/\widehat{C}$ are homeomorphic (note that $R(\widehat{\pi_C}) = (\widehat{C} \times \widehat{C}) \cup \Delta_X$ where $\widehat{\pi_C} : \widehat{X} \to \widehat{X}/\widehat{C}$ is the canonical function).

According to (1) in Theorem 5.3, any indicator topological space, that is $\widehat{X}$ for some transformation semigroup (X, S), is a T_0 Alexandroff space. The following theorem asserts that this is all we can say about the topological properties of $\widehat{X}$, without any other hypotheses.

Theorem 5.9. *A topological space A is T_0 and Alexandroff if and only if it is homeomorphic to the indicator topological space of some transformation semigroup with not necessarily compact Hausdorff phase space, however if A is finite we may find transformation semigroups with compact Hausdorff phase space.*

Proof. "$\Rightarrow$" Let $X := A$ with discrete topology and $S := \{f \in X^X : \forall x \in X xf \in V_A(x)\}$ (with discrete topology). Thus S is a topological semigroup (with composition) and (X, S) is a transformation semigroup such that $\widehat{x} = xS = V_A(x)$. Define $\varphi : A \to \widehat{X}$ with $\varphi(x) = \widehat{x}$. Since A is T_0 then φ is a bijection. Moreover for any $x \in X$ we have $\varphi(V_A(x)) = V_{\widehat{X}}(\varphi(x))$ which completes the proof.
"$\Leftarrow$" Use (1) in Theorem 5.3. $\square$

Remark 5.10. Let (X, S) be a transformation semigroup and K be the set of all closed invariant subsets of X. If $\mathrm{h}(X, S) < \infty$ then by Theorem 3.2, K is finite and closed under arbitrary union. If X be Alexandroff then again K is closed under arbitrary union. As we will see, this property has remarkable consequences.

Theorem 5.11. *Let (X, S) be a transformation semigroup in which the arbitrary union of closed invariant subsets of X is closed. Then there exists a one-to-one correspondence between closed invariant subsets of X and nonempty open subsets of $\widehat{X}$ which preserves inclusion.*

Proof. Let T be the indicator topology of X on $\widehat{X}$ and K be the set of all closed invariant subsets of X. Define $\varphi : T - \{\emptyset\} \to K$ with $\varphi(V) = \bigcup V$ $(V \in T - \{\emptyset\})$. If $V \in T - \{\emptyset\}$ then $\bigcup V = \bigcup\{\overline{xS} : \overline{xS} \in V\}$. Since the arbitrary union of closed invariant sunsets of X is closed, $\bigcup V \in K$, and $\varphi : T - \{\emptyset\} \to K$ is well-defined. On the other hand if $x \in X$ and $V \in T - \{\emptyset\}$ then $x \in \bigcup V$ if and only if $\widehat{x} \in V$, thus φ is one-to-one. Finally suppose $C \in K$ and set $V = \{\widehat{x} : x \in C\}$. We conclude from $\bigcup V_{\widehat{X}}(\widehat{x}) = \overline{xS}$ $(x \in X)$ that $V \in T - \{\emptyset\}$ and $\varphi(V) = C$. $\square$

Using the connection obtained in the previous theorem we have the following corollary which is valid for transformation semigroups with arbitrary phase space, i.e. we don't make here the restriction of being compact and Hausdorff for phase space.

Corollary 5.12. *Let (X, S) be a transformation semigroup in which arbitrary union of closed invariant sunsets of X is closed. Then we have:*

1. *$\widehat{X}$ is connected if and only if X is the only nonempty clopen invariant subset of X;*
2. *$\widehat{X}$ is paracompact if and only if for any $x \in X$ the set $\{\overline{yS} : \overline{xS} \cap \overline{yS} \neq \emptyset\}$ is finite;*
3. *$\widehat{X}$ is locally finite if and only if for any $x \in X$, $\mathrm{h}(\overline{xS}) < \infty$;*
4. *If $\widehat{X}$ is locally finite then $\widehat{X}$ is compact if and only if $\mathrm{h}(X, S) < \infty$;*
5. *$\widehat{X}$ is second countable if and only if the set $\{\overline{xS} : x \in X\}$ is countable;*
6. *$\widehat{X}$ is pseudo-metrizable if and only if X is a union of its minimal sets.*

Proof.

1. Use Theorem 5.11.
2. By [2], Theorem 2.8, $\widehat{X}$ is paracompact if and only if for any $x \in X$, $V(\widehat{x})$ meets only a finite number of $V(\widehat{x})$, which is equivalent to the fact that for any $x \in X$ the set $\{\overline{yS} : \overline{xS} \cap \overline{yS} \neq \emptyset\}$ is finite.
3. Recall that an Alexandroff space A is locally finite if and only if for any $x \in A$, $V(x)$ is finite. Now use (2) in Theorem 5.3.
4. By [2], Theorem 2.8, $\widehat{X}$ is compact if and only if $\widehat{X}$ is finite, and this equivalent to the finiteness of $\mathrm{h}(X, S)$.
5. By definition $\widehat{X} = \{\overline{xS} : x \in X\}$. Now use Theorem 2.8 of [2].

6. By [2], Theorem 2.9, a T_0 Alexandroff is pseudo-metrizable if and only if it is discrete. Now use (4) in Theorem 5.11.

□

Now we are ready to explore indicator topological space $\widehat{X}$ of transformation semigroup (X,S), when $\mathrm{h}(X,S)<\infty$.

Note 5.13. By (2) in Theorem 5.3, if $\widehat{X}=\{\widehat{x_0},\dots,\widehat{x_m}\}$ and $\widehat{x_i}$s are distinct, moreover $|V_{\widehat{X}}(\widehat{x_0})|\leq\cdots\leq|V_{\widehat{X}}(\widehat{x_m})|$, then $\mathrm{seq}(X,S)=(|V_{\widehat{X}}(\widehat{x_0})|-1,\dots,|V_{\widehat{X}}(\widehat{x_m})|-1)$, thus if two transformation semigroups have homeomorphic indicator topological spaces, then they have the same indicator sequence, however there are transformation semigroups with the same indicator sequence but non-homeomorphic indicator topological spaces, so the concept of indicator topology divides the class of all transformation semigroups with the same indicator sequence into smaller subclasses.

Note 5.14. Let us consider two simple and well-known examples of transformation semigroups:

- Let X be the unit circle in $\mathbb{R}^2$ and S be its rational rotations group, then (X,S) is minimal and like other minimal transformation semigroups $\mathrm{h}(X,S)=0$, $\mathrm{seq}(X,S)=(0)$ and $\widehat{X}$ is a singleton.
- Let $X=[0,1]$ with induced topology of $\mathbb{R}$ and S be the group of all homeomorphisms of X, then in the transformation semigroup (X,S) we have $\{\overline{xS}:x\in X\}=\{\{0,1\},X\}$, so $\mathrm{h}(X,S)=1$, $\mathrm{seq}(X,S)=(0,1)$, $\widehat{X}=\{\widehat{0},\widehat{\tfrac{1}{2}}\}$ under topology $\{\{\widehat{0}\},\widehat{X},\emptyset\}$.

Remark 5.15. In the transformation semigroup (X,S), in the definition of $\mathrm{h}(X,S)$, $\mathrm{seq}(X,S)$ (Definition 4.1), and $\widehat{X}$ (Definition 5.1), there is no need to restrict X to be compact or Hausdorff.

Using the above remark we bring the following examples omitting the hypothesis of being compact and Hausdorff about phase spaces.

Example 5.16. Let G be a topological group and H be one of its subgroups, then (G,H) is a transformation group. We have $\mathrm{h}(G,H)<\infty$ if and only if $G/\overline{H}=\{g\overline{H}:g\in G\}$ is finite, moreover in this case $\mathrm{h}(G,H)=|G/\overline{H}|-1$, $\mathrm{seq}(G,H)=(0,\dots,0)$, and the indicator topology of (G,H) is discrete.

Example 5.17. Consider left transformation group $(\mathrm{GL}(n,\mathbb{R}),\mathbb{R}^n)$ with natural action of $\mathrm{GL}(n,\mathbb{R})$ (the collection of all invertible $n\times n$ matrices with real entries) on $\mathbb{R}^n$. Since in this text we deal with right transformation semigroups, suppose $(X,S_1)=(\mathbb{R}^n,\mathrm{GL}(n,\mathbb{R}))$ with $xA:=A^{-1}x$ ($x\in\mathbb{R}^n, A\in\mathrm{GL}(n,\mathbb{R})$). We have $\overline{xS_1}=\mathbb{R}^n$ for $x\neq 0$ and $\overline{0S_1}=\{0\}$, thus $\mathrm{seq}(\mathbb{R}^n,\mathrm{GL}(n,\mathbb{R}))=(0,1)$, and $\widehat{X}=\{\{0\},\mathbb{R}^n\}$ with topology $\{\emptyset,\{\{0\}\},\widehat{X}\}$.

Example 5.18. Consider the transformation group $(\mathbb{R}^2,H)$ with

$$H=\left\{\begin{bmatrix} a & b\\ 0 & a\end{bmatrix}\in\mathrm{GL}(2,\mathbb{R}):b\in\mathbb{R},a\in(0,+\infty)\right\}$$

with action inherited from Example 5.17. We have:

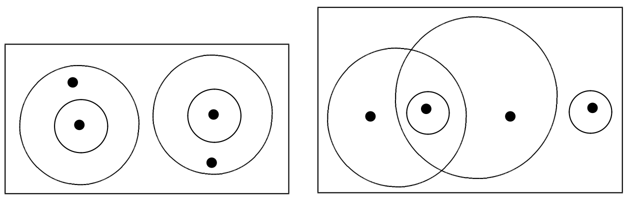
$$\overline{(x,y)H}=\begin{cases}(0,0) & x=y=0\\ [0,+\infty)\times\{0\} & x>0\wedge y=0\\ (-\infty,0]\times\{0\} & x<0\wedge y=0\\ \mathbb{R}\times[0,+\infty) & y>0\\ \mathbb{R}\times(-\infty,0] & y<0\end{cases},$$

thus $\mathrm{seq}(\mathbb{R}^2,H)=(0,1,1,3,3)$ and $\widehat{X}=\{\{(0,0)\},[0,+\infty)\times\{0\},(-\infty,0]\times\{0\},\mathbb{R}\times[0,+\infty),\mathbb{R}\times(-\infty,0]\}$ with topological basis $\{\{\{(0,0)\}\},\{[0,+\infty)\times\{0\},\{(0,0)\}\},\{(-\infty,0]\times\{0\},\{(0,0)\}\},\{\mathbb{R}\times(-\infty,0],[0,+\infty)\times\{0\},(-\infty,0]\times\{0\},\{(0,0)\}\},\{\mathbb{R}\times[0,+\infty),[0,+\infty)\times\{0\},(-\infty,0]\times\{0\},\{(0,0)\}\}\}$.

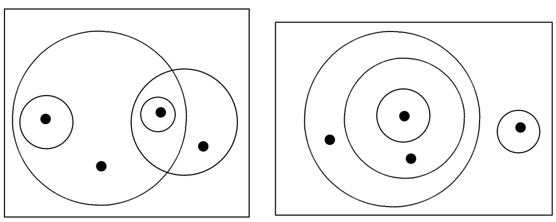
Table 5.19. All transformation semigroups like (X,S) with $\mathrm{h}(X,S) < 3$ and the same indicator sequence, have homeomorphic indicator topological spaces. But this is not true in general. See the following table for $\mathrm{h}(X,S) < 4$ (a,b,c,d are distinct):

$\mathrm{h}(X,S)$	$\mathrm{seq}(X,S)$	the minimal base of the indicator topology on $\widehat{X}$
0	(0)	$\{\{a\}\}$
1	(0,0)	$\{\{a\},\{b\}\}$
	(0,1)	$\{\{a\},\{a,b\}\}$
2	(0,0,0)	$\{\{a\},\{b\},\{c\}\}$
	(0,0,1)	$\{\{a\},\{b\},\{a,c\}\}$
	(0,0,2)	$\{\{a\},\{b\},\{a,b,c\}\}$
	(0,1,1)	$\{\{a\},\{a,b\},\{a,c\}\}$
	(0,1,2)	$\{\{a\},\{a,b\},\{a,b,c\}\}$
3	(0,0,0,0)	$\{\{a\},\{b\},\{c\},\{d\}\}$
	(0,0,0,1)	$\{\{a\},\{b\},\{c\},\{a,d\}\}$
	(0,0,0,2)	$\{\{a\},\{b\},\{c\},\{a,b,d\}\}$
	(0,0,0,3)	$\{\{a\},\{b\},\{c\},\{a,b,c,d\}\}$
	(0,0,1,1)	$\{\{a\},\{b\},\{a,c\},\{b,d\}\}$ or $\{\{a\},\{b\},\{a,c\},\{a,d\}\}$
	(0,0,1,2)	$\{\{a\},\{b\},\{a,c\},\{a,b,d\}\}$ or $\{\{a\},\{b\},\{a,c\},\{a,c,d\}\}$
	(0,0,1,3)	$\{\{a\},\{b\},\{a,c\},\{a,b,c,d\}\}$
	(0,0,2,2)	$\{\{a\},\{b\},\{a,b,c\},\{a,b,d\}\}$
	(0,0,2,3)	$\{\{a\},\{b\},\{a,b,c\},\{a,b,c,d\}\}$
	(0,1,1,1)	$\{\{a\},\{a,b\},\{a,c\},\{a,d\}\}$
	(0,1,1,2)	$\{\{a\},\{a,b\},\{a,c\},\{a,b,d\}\}$
	(0,1,1,3)	$\{\{a\},\{a,b\},\{a,c\},\{a,b,c,d\}\}$
	(0,1,2,2)	$\{\{a\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$
	(0,1,2,3)	$\{\{a\},\{a,b\},\{a,b,c\},\{a,b,c,d\}\}$

Thus for each of indicator sequences $(0,0,1,1)$ and $(0,0,1,2)$ there are two non-homeomorphic possible indicator topologies on $\widehat{X}$ (see the following diagrams).



Two possible minimal topological bases on $\widehat{X}$
with non-homeomorphic topologies, for $\mathrm{seq}(X,S) = (0,0,1,1)$



Two possible minimal topological bases on $\widehat{X}$
with non-homeomorphic topologies, for $\mathrm{seq}(X,S) = (0,0,1,2)$

6. First steps in generalization of finite height case: Indicator sequence of a transformation semigroup with pointwise finite height property

Not only "pointwise almost periodicity" of a transformation semigroup has been discussed individually in several contexts but also has interaction with some other concepts in a transformation semigroup like distality and has been studied from this point of view too [5]. It is well-known that transformation semigroup (X, S) is pointwise almost periodic if and only if it a disjoint union of its minimal sets, i.e. $\mathrm{h}(\overline{xS}, S) = 0$ for all $x \in X$ (Chapter I in [4]).
In this section we want to consider transformation semigroups with pointwise finite height, a transformation semigroup (X, S) has *pointwise finite height property* if $\mathrm{h}(\overline{xS}, S) < \infty$ for all $x \in X$.
First let us make a short comparison between pointwise almost periodicity and pointwise finite height property.

Comparison 1. *It is well-known that any factor of a pointwise almost periodic transformation semigroup is also a pointwise almost periodic transformation semigroup, i.e. any factor of a* pointwise zero height *transformation semigroup is a pointwise zero height transformation semigroup. Now we can say in a more general sense that any factor of a transformation semigroup with* pointwise finite height *property has also pointwise finite height property, this is because for onto homomorphism $\varphi : (X, S) \to (Y, S)$ and closed invariant subset Z of X, $\varphi(Z)$ is a closed invariant subset of Y and* $\mathrm{h}(\varphi(Z), S) \leq \mathrm{h}(Z, S)$ *(use Note 7 in [8]); and so for every $x \in X$ we have* $\mathrm{h}(\overline{\varphi(x)S}, S) \leq \mathrm{h}(\overline{xS}, S)$ *which leads to the desired result.*

Comparison 2. *In a transformation semigroup (X, S), (X, S) is a disjoint union of minimal closed invariant subsets if and only if it has pointwise zero height property (if and only if it is pointwise almost periodic). Now we can say in a more general sense that if (X, S) is a disjoint union of finite height closed invariant subsets, then it has pointwise finite height property.*

Although being disjoint union of closed invariant subsets with finite height implies pointwise finite height property, the following counterexample shows that the reverse implication is not valid.

Counterexample 6.1. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with induced topology of $\mathbb{R}$. Define $f_1, f_2 : X \to X$ with:

$$xf_1 = \begin{cases} \frac{2x}{2x+1} & x \in \{\frac{1}{2^n} : n \geq 1\} \\ \frac{x}{x+1} & \text{otherwise} \end{cases} \quad , \quad xf_2 = \begin{cases} 1 & x = 1 \\ \frac{x}{1-x} & \text{otherwise} \end{cases} .$$

For $i = 1, 2$ consider transformation semigroup (X, S_i) where $S_i = \{f_i^n : n \geq 0\}$. We have:

- (X, S_1) has pointwise finite height property, and it is disjoint union of its closed invariant subsets with finite height.
- (X, S_2) has pointwise finite height property, but it is not disjoint union of its closed invariant subsets with finite height.

Definition 6.2. Let (X, S) be a transformation semigroup with pointwise finite height property. We introduce the indicator sequence of (X, S), $\mathrm{seq}(X, S)$, in a natural way. Suppose $\alpha(n) := \mathrm{card}(\{\overline{xS} : x \in X, \mathrm{h}(\overline{xS}, S) = n\})$ ($n \in \mathbb{N} \cup \{0\}$). Indicator sequence of (X, S) is a sequence of (ordinal) length $\eta = \alpha(0) + \alpha(1) + \alpha(2) + \cdots (= \sup\{\alpha(0) + \cdots + \alpha(n) : n \in \omega\})$ in the form $(k(\lambda) : \lambda < \eta)$ for $k(\lambda) = m$ whenever $\alpha(0) + \cdots + \alpha(m-1) \leq \lambda < \alpha(0) + \cdots + \alpha(m-1) + \alpha(m)$ ($m \in \mathbb{N} \cup \{0\}, \alpha(-1) := -1$).

Remark 6.3. One can easily see that Definition 6.2 is a generalization of Definition 4.1.

Proposition 6.4. *In the transformation semigroup (X, S) with pointwise finite height property, if Z, W are nonempty closed invariant subsets of (X, S), then:*

- *If $Z \subseteq W$, then* $\mathrm{seq}(Z, S)$ *is a subsequence of* $\mathrm{seq}(W, S)$.

- *If* $\mathrm{seq}(X, S) = (k(\lambda) : \lambda < \eta)$ *and* $\beta < \lambda < \eta$*, then* $k(\beta) \leq k(\lambda) \leq \lambda$.
- *If* $\mathrm{seq}(X, S) = (k(\lambda) : \lambda < \eta)$ *and* $\mathrm{seq}(Y, T) = (d(\theta) : \theta < \rho)$*, then* $\mathrm{seq}(X \times Y, S \times T)$ *is a rearrangement of* $((k(\lambda) + 1)(d(\theta) + 1) - 1 : \lambda < \eta, \theta < \rho)$

Proof. It is clear by Definition 6.2, Theorem 4.4 and Theorem 4.5. □

Proposition 6.5. *In the transformation semigroup* (X, S) *with pointwise finite height property, infinite countable indicator topological space and* $\mathrm{seq}(X, S) = (k(\lambda) : \lambda < \eta)$*, we have:*

1. $\eta \leq \omega\omega$.
2. *The following statements are equivalent:*
   - η *is an infinite cardinal number.*
   - $\eta = \omega$.
   - $\forall k \in \omega \ (k \geq \min(\{n \in \omega : \alpha(n) = \omega\} \cup \{\omega\}) \Rightarrow \alpha(k) = 0)$.
3. *if* $\eta = \omega p + q$ *whenever* $q < \omega$*, then:*
   $$p - 1 \leq \mathrm{card}(\{n \geq 0 : \alpha(n) \text{ is infinite}\}) \leq p.$$
4. $\eta = \omega\omega$ *if and only if there exists an infinite subset* $\{n_l : l \in \omega\}$ *in* ω *such that* $\alpha(n_l)$ *is infinite for all* $l \in \omega$.

Proof.

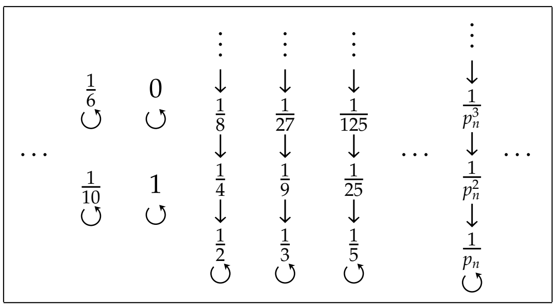
1. For each $n \in \omega$, $\alpha(n) \leq \omega$ (since indicator topological space of (X, S), i.e. $\{\overline{xS} : x \in X\}$, is countable), thus using induction on n we have $\alpha(0) + \cdots + \alpha(n) \leq \overbrace{\omega + \cdots + \omega}^{n+1 \text{ times}} = \omega(n + 1) \leq \omega\omega$, so $\eta \leq \omega\omega$, which leads to the desired result.
2. Since the indicator topological space of (X, S) is infinite, $\eta \geq \omega$. By (1) we have $\eta \leq \omega\omega$. ω is the unique cardinal number η satisfying $\omega \leq \eta \leq \omega\omega$.

□

Note 6.6. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with induced topology of $\mathbb{R}$. Then for every $\eta \leq \omega\omega$ there exist a continuous map $f : X \to X$ and a sequence $(k(\lambda) : \lambda < \eta)$ such that in transformation semigroup (X, S) for $S = \{f^n : n \geq 0\}$ we have $\mathrm{seq}(X, S) = (k(\lambda) : \lambda < \eta)$.

Proof. We distinguish the following cases (suppose p_n is the n'th prime number):

- $\eta = \omega\omega$. Define $f : X \to X$ with:
  $$xf = \begin{cases} \frac{1}{p_n^k} & x = \frac{1}{{p_n}^{k+1}} \wedge k \geq 1 \\ x & \text{otherwise} \end{cases}.$$
  which can be easily found in the following diagram:



So:

$$\overline{xS} = \begin{cases} \{\frac{1}{p_n^i} : 1 \leq i \leq k+1\} & x = \frac{1}{{p_n}^{k+1}} \wedge k \geq 1 \\ \{x\} & \text{otherwise} \end{cases},$$

$$\mathrm{h}(\overline{xS}, S) = \begin{cases} k & x = \frac{1}{{p_n}^{k+1}} \wedge k \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

which shows $\alpha(n) = \omega$ for all $n \in \omega$ and $\eta = \omega\omega$.

- $\eta = \omega q + r$ with $q, r \in \omega$. Define $f : X \to X$ with:

$$xf = \begin{cases} \frac{1}{p_n^k} & x = \frac{1}{{p_n}^{k+1}} \wedge 1 \leq k \leq q-1 \\ \frac{1}{p_{n+1}^k} & x = \frac{1}{{p_n}^k} \wedge q+1 \leq k \leq q+r \wedge n \leq q+1 \\ x & \text{otherwise} \end{cases}.$$

So:

$$\overline{xS} = \begin{cases} \{\frac{1}{p_n^i} : 1 \leq i \leq k+1\} & x = \frac{1}{{p_n}^{k+1}} \wedge 1 \leq k \leq q-1 \\ \{\frac{1}{p_j^k} : n \leq j \leq q+1\} & x = \frac{1}{{p_n}^k} \wedge q+1 \leq k \leq q+r \wedge n \leq q+1 \\ \{x\} & \text{otherwise} \end{cases},$$

and $\mathrm{h}(\overline{xS}, S) = |\overline{xS}| - 1$ which shows $\alpha(n) = \omega$ for $n \in \{0, ..., q-1\}$, $\alpha(q) = r$, $\alpha(n) = 0$ for $n > q$, and $\eta = \omega q + r$.

□

Example 6.7. Consider the transformation group $(X, S_2) := (\{x \in \mathbb{R}^n : \|x\| \leq 1\}, \mathrm{O}(n))$, where $\mathrm{O}(n)$ is the set of all orthogonal elements of $\mathrm{GL}(n, \mathbb{R})$, with action inherited from Example 5.17. For $D_r = \{x \in \mathbb{R}^n : \|x\| = r\}$ ($r \in [0, 1]$), we have $\overline{xS_2} = D_r$ for any $x \in D_r$, thus $\mathrm{seq}(X, \mathrm{O}(n, \mathbb{R})) = (k(\lambda) : \lambda < c)$ where $k(\lambda) = 0$ for all $\lambda < c$ and $c = \mathrm{card}(\mathbb{R})$. Moreover $\widehat{X} = \{D_r : r \in [0, 1]\}$ with discrete topology.

In the following example we omit the hypothesis of compactness for phase space.

Example 6.8. Let V be a discrete topological vector space over field F. One may consider transformation group (V, F), where $vr := rv$ ($v \in V, r \in F$). Then $\mathrm{seq}(V, F) = (k(\lambda) : \lambda < \mathrm{card}(V/F))$, where $k(0) = 0$ and $k(\lambda) = 1$ for $\lambda > 0$. Moreover $\widehat{V} = \{Fv : v \in V\}$ has topological basis $\{\{\{0\}, Fv\} : v \in V\}$.

As it has been mentioned in Lemma 12 of Chapter I in [4], in transformation group (X, T) for $x \in X$, $t \in T$ and S a normal subgroup of T we have "x is an almost periodic point of (X, S) if and only if xt is an almost periodic point in (X, S)", i.e. $\mathrm{h}(\overline{xS}, S) = 0 \Leftrightarrow \mathrm{h}(\overline{xtS}, S) = 0$. In the following lemma we will see $\mathrm{h}(\overline{xS}, S) = \mathrm{h}(\overline{xtS}, S)$.

Lemma 6.9. *In the transformation group (X, G) suppose $t \in G$, $x \in X$ and S is a normal subgroup of G. Then we have* $\mathrm{h}(\overline{xS}, S) = \mathrm{h}(\overline{xtS}, S)$.

Proof. Suppose $\mathrm{h}(\overline{xtS}, S) < \infty$. We have:

$$\begin{aligned} \mathrm{h}(\overline{xtS}, S) + 1 &= |\{\overline{yS} : y \in \overline{xtS}\}| \quad \text{(use Theorem 3.2)} \\ &= |\{\overline{yS} : y \in \overline{xSt}\}| \quad \text{(since } S \text{ is normal in } G) \\ &= |\{\overline{yS} : yt^{-1} \in \overline{xS}\}| \\ &= |\{\overline{yt^{-1}St} : yt^{-1} \in \overline{xS}\}| \quad \text{(since } S \text{ is normal in } G) \\ &= |\{\overline{zSt} : z \in \overline{xS}\}| = |\{\overline{zS} : z \in \overline{xS}\}| \\ &= \mathrm{h}(\overline{xS}, S) + 1 \quad \text{(use Theorem 3.2).} \end{aligned}$$

□

Note 6.10. In the transformation group (X, G) suppose $t \in G$, A is a nonempty subset of X and S is a normal subgroup of G. Then we have $\mathrm{h}(\overline{AS}, S) = \mathrm{h}(\overline{AtS}, S)$.

Proof. Use a similar method described in Lemma 6.9. □

Note 6.11. In the transformation group (X, G) suppose S is a normal subgroup of G and $x \in X$. Then $\max(\mathrm{h}(\overline{xG}, G), \mathrm{h}(\overline{xS}, S)) \leq \mathrm{h}(\overline{xG}, S)$

Proof. Any G invariant subset of $\overline{xG}$ is an S invariant subset of $\overline{xG}$, thus $\mathrm{h}(\overline{xG}, G) \leq \mathrm{h}(\overline{xG}, S)$. Moreover $\overline{xS} \subseteq \overline{xG}$, which leads to $\mathrm{h}(\overline{xS}, S) \leq \mathrm{h}(\overline{xG}, S)$. □

Lemma 6.12. *In the transformation group (X, G) suppose S is a normal subgroup of G, with $[G : S] = n \in \mathbb{N}$ and $x \in X$. We have $\mathrm{h}(\overline{xG}, S) \leq (n-1) + n\mathrm{h}(\overline{xS}, S)$.*

Proof. Since $[G : S] = n$, there exist $t_1, ..., t_n \in G$ such that G is disjoint union of $t_1S, ..., t_nS$, therefore $\overline{xG} = \bigcup_{1 \leq i \leq n} \overline{xt_iS}$. So by Lemma 8 in [8] and Lemma 6.9 we have:

$$\mathrm{h}(\overline{xG}, S) \leq (n-1) + \sum_{1 \leq i \leq n} \mathrm{h}(\overline{xt_iS}, S) = (n-1) + n\mathrm{h}(\overline{xS}, S).$$

□

Lemma 6.13. *In the transformation group (X, G) suppose S is a normal subgroup of G, with $[G : S] = n \in \mathbb{N}$ and A is a nonempty closed $G-$invariant subset of X, then $\mathrm{h}(A, S) \leq (n-1) + n\mathrm{h}(A, G)$.*

Proof. Choose $t_1, ..., t_n \in G$ as in proof of Lemma 6.12. There exists $\Gamma \subseteq A$ with $|\Gamma| = \mathrm{h}(A, G) + 1$ such that $\{\overline{xG} : x \in A\} = \{\overline{xG} : x \in \Gamma\}$. It is clear that $\{\overline{xt_iS} : x \in \Gamma, i \in \{1, ..., n\}\} \subseteq \{\overline{xS} : x \in A\}$. Suppose $y \in A$. There exists $x \in \Gamma$ such that $\overline{yG} = \overline{xG} = \overline{xt_1S} \cup \cdots \cup \overline{xt_nS}$, thus there exists $k \in \{1, ..., n\}$ such that $y \in \overline{xt_kS}$, thus $\overline{yS} \subseteq \overline{xt_kS}$. If $\overline{yS} \neq \overline{xt_kS}$, then for any $i \in \{1, ..., n\}$, $\overline{ySt_i}$ is a proper subset of $\overline{xt_kSt_i}$, thus $\overline{yG} = \overline{yt_1S} \cup \cdots \cup \overline{yt_nS} = \overline{ySt_1} \cup \cdots \cup \overline{ySt_n}$ is a proper subset of $\overline{xt_kSt_1} \cup \cdots \cup \overline{xt_kSt_n} = \overline{xt_kG} = \overline{xG}$, which is a contradiction, therefore $\overline{yS} = \overline{xt_kS}$. Hence $\overline{yS} \in \{\overline{xt_iS} : x \in \Gamma, i \in \{1, ..., n\}\}$ for any $y \in A$, which shows $\{\overline{yS} : y \in A\} = \{\overline{xt_iS} : x \in \Gamma, i \in \{1, ..., n\}\}$ which leads to $\mathrm{h}(A, S) + 1 \leq n(\mathrm{h}(A, G) + 1)$. □

As it has been mentioned in Theorem 13 of Chapter I in [4], in transformation group (X, T) for $x \in X$, $t \in T$ and S a normal syndetic subgroup of T we have "(X, S) is pointwise almost periodic if and only if (X, T) is pointwise almost periodic", i.e. "(X, S) has pointwise zero height property if and only if (X, T) has pointwise zero height property". Moreover we know that if $[T : S] < \infty$, then S is a syndetic subgroup of T, also whenever T is discrete $[T : S] < \infty$ if and only if S is a syndetic subgroup of T. We have the following similar Theorem for pointwise finite height property:

Theorem 6.14. *In the transformation group (X, G) suppose S is a normal subgroup of G with $[G : S] = n \in \mathbb{N}$. (X, S) has pointwise finite height property if and only if (X, G) has pointwise finite height property.*

Proof. Using Lemma 6.12 and Note 6.11 for any $x \in X$ we have $\mathrm{h}(\overline{xG}, G) \leq \mathrm{h}(\overline{xG}, S) \leq (n-1) + n\mathrm{h}(\overline{xS}, S)$, therefore if (X, S) has pointwise finite height property, then (X, G) has pointwise finite height property.
On the other hand using Lemma 6.13 and Note 6.11 for any $x \in X$ we have $\mathrm{h}(\overline{xS}, S) \leq \mathrm{h}(\overline{xG}, S) \leq (n-1) + n\mathrm{h}(\overline{xG}, G)$, therefore if (X, G) has pointwise finite height property, then (X, S) has pointwise finite height property. □

7. A glance to: Indicator sequence and indicator topological space's approach and universality

In this section we deal with universal objects of some subclasses of the class of all transformation semigroups with discrete infinite phase semigroup S, specially we are interested in subclasses consist of transformation semigroups with a "given finite height", a "given finite indicator sequence", or a "given finite indicator topological space ".
We recall that all phase spaces are compact Hausdorff and all phase semigroups are discrete.
For most of the constructions of this section the following remark is essential.

Remark 7.1. It is well-known that for any discrete topological semigroup S, $(\beta S, S)$ is universal point transitive transformation semigroup in the class of all point transitive transformation semigroups with phase semigroup S (see, e.g. [5]), in addition, for any minimal right ideal I of βS, (I, S) is a universal minimal transformation semigroup in the class of all minimal transformation semigroups with phase semigroup S, so (I, S) is universal in the class of all transformation semigroups with height 0, compact Hausdorff phase space and discrete phase semigroup S.

Construction 7.2. *Let $C_m(S)$ denote the class of all transformation semigroups with compact Hausdorff phase space, discrete phase semigroup S and finite height $m \in \mathbb{N} \cup \{0\}$. Then the transformation semigroup $(\{1, ..., m+1\} \times \beta S, S)$, with $(i, p)s := (i, ps)$ $(i \in \{1, ..., m+1\}, p \in \beta S, s \in S)$ is universal for $C_m(S)$.*

Proof. Suppose $(X, S) \in C_m(S)$. By Theorem 3.2, there exist $x_1, ..., x_{m+1} \in X$ such that $\{\overline{xS} : x \in X\} = \{\overline{x_i S} : i \in \{1, ..., m+1\}\}$ (therefore $X = \bigcup\{\overline{x_i S} : i \in \{1, ..., m+1\}\}$). The map $\varphi : (\{1, ..., m+1\} \times \beta S, S) \to (X, S)$ with $\varphi(i, p) = x_i p$ $(i \in \{1, ..., m+1\}, p \in \beta S)$ is an onto homomorphism. □

Corollary 7.3. *Let Y be a finite T_0 topological space. Let $\mathcal{T}_Y(S)$ denote the class of all transformation semigroups with compact Hausdorff phase space, discrete phase semigroup S and indicator topological space (homeomorphic to) Y, whenever $|Y| = m+1 \in \mathbb{N}$, since $\mathcal{T}_Y(S)$ is a subclass of $C_m(S)$ (Construction 7.2), $(\{1, ..., m+1\} \times \beta S, S)$, with $(i, p)s := (i, ps)$ $(i \in \{1, ..., m+1\}, p \in \beta S, s \in S)$ is universal for $\mathcal{T}_Y(S)$.*

Corollary 7.4. *Let $C_m^{(n_0,...,n_m)}(S)$ be the class of all transformation semigroups with compact Hausdorff phase space, discrete phase semigroup S and indicator sequence $(n_0, ..., n_m)$. Since $C_m^{(n_0,...,n_m)}(S)$ is a subclass of $C_m(S)$ (Construction 7.2), $(\{1, ..., m+1\} \times \beta S, S)$, with $(i, p)s := (i, ps)$ $(i \in \{1, ..., m+1\}, p \in \beta S, s \in S)$ is universal for $C_m^{(n_0,...,n_m)}(S)$.*

Construction 7.5. *Let $C_{\mathrm{fin}}(S)$ be the class of all transformation semigroups with compact Hausdorff phase space, discrete phase semigroup S and finite height. Suppose for each $m \in \mathbb{N} \cup \{0\}$, (W_m, S) is a universal transformation semigroup for $C_m(S)$ (Construction 7.2). The transformation semigroup $(\prod_{m \in \mathbb{N} \cup \{0\}} W_m, S)$ is a universal transformation semigroup for $C_{\mathrm{fin}}(S)$.*

The following construction is similar to Auslander's method for describing universal minimal sets in [3].

Construction 7.6. *Suppose $C_\eta^{(k(\lambda):\lambda<\eta)}(S)$ is the class of all pointwise finite height transformation semigroups with compact Hausdorff phase space, discrete phase semigroup S and indicator sequence $(k(\lambda) : \lambda < \eta)$. Since for any $(X, S) \in C_\eta^{(k(\lambda):\lambda<\eta)}(S)$ we have $\mathrm{card}(X) \leq \mathrm{card}(\beta S)\mathrm{card}(\eta)$, there exists a set $A = \{(X_\alpha, S) : \alpha \in \Gamma\}$ such that any element of $C_\eta^{(k(\lambda):\lambda<\eta)}(S)$ is isomorphic to one of the elements of A. The transformation semigroup $(\prod_{\alpha \in \Gamma} X_\alpha, S)$ is a universal transformation semigroup for $C_\eta^{(k(\lambda):\lambda<\eta)}(S)$.*

Note 7.7. We have:

- Regarding Corollary 7.4, for some subclasses like $C_m^{(0,...,0)}(S)$, all of the universal elements of $C_m^{(0,...,0)}(S)$ which belongs to $C_m^{(0,...,0)}(S)$ are isomorphic to $(\{1, ..., m+1\} \times I, S) \in C_m^{(0,...,0)}(S)$ where I is an arbitrary minimal right ideal of βS and $(\{1, ..., m+1\} \times I, S)$ considered as a sub-transformation semigroup of $(\{1, ..., m+1\} \times \beta S, S)$.

- Regarding Corollary 7.4, for some subclasses like $C_m^{(0,0,1,1,n_4,\ldots,n_m)}(S)$ none of its universal objects belongs to $C_m^{(0,0,1,1,n_4,\ldots,n_m)}(S)$ (use Table 5.19).
- Regarding Construction 7.5, none of the universal objects for $C_{\text{fin}}(S)$ belongs to $C_{\text{fin}}(S)$. Since for any $m \in \mathbb{N}$, there exists a transformation semigroup $(X, S) \in C_{\text{fin}}(S)$ with $\text{h}(X, S) = m$. If (Z, S) is universal for $C_{\text{fin}}(S)$, then there exists an onto homomorphism $\varphi : (Z, S) \to (X, S)$, so $\text{h}(Z, S) \geq \text{h}(X, S) = m$. Thus $\text{h}(Z, S) \geq m$ for all $m \in \mathbb{N}$ and $\text{h}(Z, S) = \infty$, which shows $(Z, S) \notin C_{\text{fin}}(S)$.

8. A short overview

Let C denote the class of all transformation semigroups, then C is a disjoint union of the classes $C_\infty, C_0, C_1, \ldots$, where C_m is consisted of the elements of C like (X, S) with $\text{h}(X, S) = m$ (see the following diagram).

C

C_0	C_1	C_2	$\cdots$	C_∞

If $m < \infty$, for every $(X, S) \in C_m$ we assign a unique sequence $\text{seq}(X, S) = (n_0, \ldots, n_m)$ of nonnegative integers with $0 \leq n_0 \leq n_1 \leq \cdots \leq n_m$ and $n_i \leq i$ $(0 \leq i \leq m)$. Let $B_m = \{(n_0, \ldots, n_m) : 0 \leq n_0 \leq n_1 \leq \cdots \leq n_m, n_i \leq i\ (0 \leq i \leq m)\}$ (note that B_m is a totally ordered set under lexicographic ordering). Now we have a partition of C_m to $|B_m|$ (= $m+1$th Catalan number (Theorem 4.7)) of its disjoint subclasses (see the following diagram, where for $(n_0, \ldots, n_m) \in B_m$, $C_m^{(n_0,\ldots,n_m)}$ is the subclass of C_m consisting of all transformation semigroup $(X, S) \in C_m$ with $\text{h}(X, S) = (n_0, \ldots, n_m)$)

C_m

$C_m^{(0,\ldots,0)}$	$C_m^{(0,\ldots,0,1)}$	$\cdots$	$C_m^{(0,1,\ldots,m)}$

For $m < \infty$, $(X, S) \in C_m$ and $\text{seq}(X, S) = (n_0, \ldots, n_m)$ we assign to (X, S) its indicator topological space: $\widehat{X}$, a T_0 topological space. In this way we have a partition of $C_m^{(n_0,\ldots,n_m)}$ to some of its disjoint subclasses (the subclasses of transformation semigroups with homeomorphic indicator topology) (see the following diagram in the case of $(n_0, \ldots, n_m) = (0, 0, 1, 1, 1, 1)$, where $a_0, a_1, a_2, a_3, a_4, a_5$ are distinct).

$C_5^{(0,0,1,1,1,1)}$

class of all transformation semigroups in $C_5^{(0,0,1,1,1,1)}$ with indicator topological space generated by a basis in the form of $\{\{a_0\}, \{a_1\}, \{a_0, a_2\}, \{a_0, a_3\}, \{a_0, a_4\}, \{a_0, a_5\}\}$
class of all transformation semigroups in $C_5^{(0,0,1,1,1,1)}$ with indicator topological space generated by a basis in the form of $\{\{a_0\}, \{a_1\}, \{a_1, a_2\}, \{a_0, a_3\}, \{a_0, a_4\}, \{a_0, a_5\}\}$
class of all transformation semigroups in $C_5^{(0,0,1,1,1,1)}$ with indicator topological space generated by a basis in the form of $\{\{a_0\}, \{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_0, a_4\}, \{a_0, a_5\}\}$

Thus we can see the following classifications of C:

1. subclasses of all elements of C with the same height;
2. subclasses of all elements of C with the same finite height and the same indicator sequence;
3. subclasses of all elements of C with the same indicator topology (so with the same height, and if this height is finite then they will have the same indicator sequence too).

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