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SEMI-FREDHOLM OPERATORS AND PERTURBATION FUNCTIONS

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Abstract. In this paper we give several remarks on [3] and a different proof of the inequalities in [8]. Also we introduce the concept of a lower perturbation function, and prove that some usual measures of noncompactness of operators and also some measures of non-strict-cosingularity of operators are lower perturbation functions. For each lower perturbation function δ, we define functions ∇_δ and K_δ, and give the connection with lower semi-Fredholm operators.

1. Introduction and preliminaries

In this paper X, Y and Z are complex Banach spaces, $B(X,Y)$ (respectively $K(X,Y)$) the set of all bounded (respectively compact) linear operators from X into Y. We shall write $B(X)$ ($K(X)$) instead of $B(X,X)$ ($K(X,X)$).

An operator $T \in B(X,Y)$ is in $\Phi_+(X,Y)$ ($\Phi_-(X,Y)$) if the range $R(T)$ is closed and the dimension $\alpha(T)$ of the null space $N(T)$ of T is finite (the codimension $\beta(T)$ of $R(T)$ in Y is finite). Operators in $\Phi_+(X,Y) \cup \Phi_-(X,Y)$ are called semi-Fredholm operators. For such operators the index, $i(T)$, is defined by $i(T) = \alpha(T) - \beta(T)$. We set $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$. The operators in $\Phi(X,Y)$ are called Fredholm. We shall write $\Phi_+(X)$ (resp. $\Phi_-(X)$, $\Phi(X)$) instead of $\Phi_+(X,X)$ (resp. $\Phi_-(X,X)$, $\Phi(X,X)$).

Recall that the essential spectral radius of $T \in B(X)$, $r_e(T)$, is defined by

$$r_e(T) = \max\{|\lambda| : T - \lambda I \notin \Phi(X)\}.$$

Let B_X denote the closed unit ball of X. Let $T \in B(X,Y)$ and

$$m(T) = \inf\{\|Tx\| \, : \, \|x\| = 1\}$$

be the *minimum modulus* of T, and let

$$n(T) = \sup\{\epsilon \geq 0 \, : \, \epsilon B_Y \subset TB_X\}$$

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be the *surjection modulus* of T. Recall that $m(T^*) = n(T)$, where $T^* \in B(Y^*, X^*)$ is the adjoint operator.

If M is a subspace of X, then J_M will denote the embedding map of M into X, and if V is a subspace of Y, then Q_V will denote the canonical map of Y onto the quotient space Y/V.

An operator $T \in B(X, Y)$ is *strictly singular* ($T \in S(X, Y)$) if, for every infinite dimensional (closed) subspace M of X, the restriction of T to M, $T|_M$, is not a homeomorphism, i.e. $m(T|_M) = 0$. An operator $T \in B(X, Y)$ is *strictly cosingular* ($T \in CS(X, Y)$) if, for every infinite codimensional closed subspace V of Y the composition $Q_V T$ is not surjective.

For $A \in B(X, Y)$, set

$$\|A\|_C = \inf\{\|A + K\| \, : \, K \in K(X, Y)\}.$$

If Ω is a non-empty bounded subset of X, then the Kuratowski measure of noncompactness of Ω is denoted by $\alpha(\Omega)$, and

$$\alpha(\Omega) = \inf\{\epsilon > 0 \, : \, \Omega \subset \cup_{i=1}^{n} D_i, D_i \subset X, \ \mathrm{diam} D_i \leq \epsilon\}.$$

For $A \in B(X, Y)$ the Kuratovski measure of noncompactness of A is denoted by $\|A\|_\alpha$ and defined by

$$\|A\|_\alpha = \inf\{k \geq 0 \, : \, \alpha(A\Omega) \leq k\alpha(\Omega), \, \Omega \subset X \text{ is bounded}\}.$$

It is easy to see that $\|A\|_\alpha = \sup\{\alpha(A\Omega) \, : \, \Omega \subset X, \, \alpha(\Omega) = 1\}$.

If Ω is a non-empty bounded subset of X, then the Hausdorff measure of noncompactness of Ω, is denoted by $q(\Omega)$, and

$$q(\Omega) = \inf\{\epsilon > 0 \, : \, \Omega \text{ has a finite } \epsilon\text{-net in } X\}.$$

For $A \in B(X, Y)$ the Hausdorff measure of noncompactness of A is denoted by $\|A\|_q$ and defined by

$$\|A\|_q = \inf\{k \geq 0 \, : \, q_Y(A\Omega) \leq k q_X(\Omega), \, \Omega \subset X \text{ is bounded}\}.$$

Recall that $\|A\|_q = q_Y(AB_X)$.

For $A \in B(X, Y)$, set (see [6])

$$\|A\|_\mu = \inf\{\|A|_L\| \, : \, L \text{ subspace of } X, \ \mathrm{codim} L < \infty\},$$

and

$$\Gamma_M(A) = \inf_{N \subset M} \|A|_N\|, \quad \Gamma(A) = \Gamma_X(A),$$

$$\Delta_M(A) = \sup_{N \subset M} \Gamma_N(A), \quad \Delta(A) = \Delta_X(A),$$

where M, N denote infinite dimensional subspaces of X (see [9]). Schechter [9] proved that Δ is a submultiplicative seminorm and

$$\Delta(A) = 0 \Longleftrightarrow A \in S(X,Y). \tag{1.1}$$

For $A \in B(X,Y)$, set (see [12])

$$K_V(A) = \inf_{W \supset V} \|Q_W A\|\,, \quad K(A) = K_{\{0\}}(A)\,,$$
$$\nabla_V(A) = \sup_{W \supset V} K_W(A)\,, \quad \nabla(A) = \nabla_{\{0\}}(A)\,,$$

where V, W denote closed infinite codimensional subspaces of Y.
Recall that [12],

$$\begin{aligned} K(A) > 0 &\Longleftrightarrow A \in \Phi_-(X,Y), \\ \nabla(A) = 0 &\Longleftrightarrow A \in CS(X,Y). \end{aligned} \tag{1.2}$$

Zemánek [13] considered the following functions

$$u(A) = \sup\{m(A|_W) \,:\, W \text{ is closed subspace of } X \text{with } \dim W = \infty\},$$
$$v(A) = \sup\{n(Q_V A) \,:\, V \text{ is closed subspace of } Y \text{with } \operatorname{codim} V = \infty\}.$$

From the definition of the strictly singular and strictly cosigular operators it is obvious that

$$u(A) = 0 \Longleftrightarrow A \in S(X,Y), \tag{1.3}$$
$$v(A) = 0 \Longleftrightarrow A \in CS(X,Y). \tag{1.4}$$

2. Results

Recall that F. Galaz-Fontes [3] introduced and investigated a perturbation function.

Definition 2.1. A perturbation function is a function γ, assigning to each pair of complex Banach spaces X, Y, and $T \in B(X,Y)$ a nonnegative number $\gamma(T)$, with the following properties:

$$\gamma(\lambda T) = |\lambda|\gamma(T),\ \lambda \in \mathbb{C}, \tag{2.1.1}$$
$$\gamma(T+K) = \gamma(T),\ K \in K(X,Y), \tag{2.1.2}$$
$$\gamma(T) \le \|T\|, \tag{2.1.3}$$
$$m(T) \le \gamma(T), \tag{2.1.4}$$
$$\gamma(T|_M) \le \gamma(T), \tag{2.1.5}$$

where M denotes an infinite dimensional subspace of X.

He proved that the quantities $\|\cdot\|_C$, $\|\cdot\|_\alpha$, $\|\cdot\|_q$, $\|\cdot\|_\mu$, Δ and u are perturbation functions.

Let us note that the following quantity is a measure of non-strict-singularity and we shall show that it is a perturbation function.

Example 2.2. For $T \in B(X,Y)$, set

$$\|T\|_S = \inf\{\|T+S\| : S \in S(X,Y)\}.$$

(2.1.1)-(2.1.3) follow easily from the definition.

Since Δ is a seminorm which annihilates precisely on the strictly singular operators [9] and such that $\Delta(T) \leq \|T\|$, $T \in B(X,Y)$, it follows that

$$\Delta(T) = \Delta(T+S) \leq \|T+S\|, \text{ for each } S \in S(X,Y).$$

Therefore

$$\Delta(T) \leq \|T\|_S.$$

Since $m(T) \leq u(T) \leq \Delta(T)$, we get $m(T) \leq \|T\|_S$. (2.1.5) follows from the fact that a restriction of a strictly singular operator to an infinite dimensional subspace of X is a strictly singular operators also.

Recall that for a given perturbation function γ, and $T \in B(X,Y)$, F. Galaz-Fontes [3] defined

$$\Gamma_{\gamma,M}(T) = \inf\{\gamma(T|_V) : V \subset M\}, \quad \Gamma_\gamma(T) = \Gamma_{\gamma,X}(T);$$
$$\Delta_{\gamma,M}(T) = \sup\{\Gamma_{\gamma,V}(T) : V \subset M\}, \quad \Delta_\gamma(T) = \Delta_{\gamma,X}(T),$$

where M, V denote closed infinite dimensional subspaces of X.

In the following γ is a perturbation function. F. Galaz-Fontes proved that if $T \in S(X,Y)$, then $\Delta_\gamma(T) = 0$ [3, Proposition 9]. Actually, we shall prove that the equivalence holds.

Proposition 2.3. *Let $T \in B(X,Y)$. Then*

$$T \in S(X,Y) \Longleftrightarrow \Delta_\gamma(T) = 0. \tag{2.3.1}$$

Proof. First we show:

$$u(T) \leq \Delta_\gamma(T) \leq \Delta(T). \tag{2.3.2}$$

From (2.1.3) and the definitions of Δ and Δ_γ it follows that $\Delta_\gamma(T) \leq \Delta(T)$.

Let M be a closed infinite dimensional subspace of X. We have

$$\gamma(T|_V) \geq m(T|_V) \geq m(T|_M),$$

for each closed infinite dimensional subspace V of M. It implies

$$\Gamma_{\gamma,M}(T) = \inf_{V\subset M} \gamma(T|_V) \geq m(T|_M).$$

Hence

$$\Delta_\gamma(T) = \sup_M \Gamma_{\gamma,M}(T) \geq \sup_M m(T|_M) = u(T).$$

(2.3.1) follows directly from (2.3.2), (1.1) and (1.3). $\square$

Let us remark that if γ is a submultiplicative seminorm then it can be proved (analogously as in [9]) that Δ_γ is a submultiplicative seminorm and

$$\Gamma_\gamma(T+S) \leq \Gamma_\gamma(T) + \Delta_\gamma(S), \quad T,\ S \in B(X,Y).$$

If $\gamma = \|\cdot\|_q$ we shall write Γ_q (Δ_q) instead of Γ_γ (Δ_γ). Analogously, we introduce Γ_α (Δ_α), Γ_μ (Δ_μ), Γ_C (Δ_C), Γ_u (Δ_u) and Γ_Δ (Δ_Δ). By [3, Lemma 8] it follows that $\Gamma_\Delta = \Gamma$ and $\Delta_\Delta = \Delta$.

Lemma 2.4. *If there exists a constant $c > 0$ such that $\gamma(T) \geq c\Delta(T)$, for each $T \in B(X,Y)$, then*

$$\Gamma_\gamma(T) \leq \Gamma(T) \leq c^{-1}\Gamma_\gamma(T), \tag{2.4.1}$$

$$\Delta_\gamma(T) \leq \Delta(T) \leq c^{-1}\Delta_\gamma(T),\ T \in B(X,Y). \tag{2.4.2}$$

Proof. (2.4.1). From (2.1.3) and the definitions of Γ and Γ_γ it follows that $\Gamma_\gamma(T) \leq \Gamma(T)$. Further from the hypothesis and [3, Lemma 8] we get

$$\Gamma_\gamma(T) \geq c\Gamma_\Delta(T) = c\Gamma(T).$$

(2.4.2) can be proved analogously to (2.4.1). $\square$

Recall that

$$\begin{aligned}
&\|T\|_\mu \geq \Delta(T) \ [9,\ \text{Theorem } 2.10],\\
&\|T\|_q \geq 2^{-1}\|T\|_\mu \geq 2^{-1}\Delta(T) \ [6,\ \text{Theorem } 3.1],\\
&\|T\|_\alpha \geq 2^{-1}\|T\|_q \geq 4^{-1}\Delta(T),\\
&\|T\|_C \geq \|T\|_\mu \geq \Delta(T),\\
&\|T\|_S \geq \Delta(T),\ T \in B(X,Y).
\end{aligned}$$

By Lemma 2.4 we get

$$\begin{aligned}
&\Gamma_\mu(T) = \Gamma(T), \quad \Delta_\mu(T) = \Delta(T),\\
&\Gamma_q(T) \leq \Gamma(T) \leq 2\Gamma_q(T), \quad \Delta_q(T) \leq \Delta(T) \leq 2\Delta_q(T),\\
&\Gamma_\alpha(T) \leq \Gamma(T) \leq 4\Gamma_\alpha(T), \quad \Delta_\alpha(T) \leq \Delta(T) \leq 4\Delta_\alpha(T),\\
&\Gamma_C(T) = \Gamma(T), \quad \Delta_C(T) = \Delta(T),\\
&\Gamma_S(T) = \Gamma(T), \quad \Delta_S(T) = \Delta(T).
\end{aligned} \tag{2.5}$$

The inequalities in (2.5) were proved in [8] and [5, Proposition 3.9]. However, our proof is different from the proofs in [8] and [5].

Remark 2.6. Clearly, $\Delta_\gamma \leq \gamma$, and by (2.3.2), for $\gamma = u$, we get

$$\Delta_u = u.$$

Recall that [3, Lemma 11]

$$B \leq \Gamma_u \leq \Gamma,$$

where $B(T) = \sup\{m(T|_V) : V$ closed subspace of X, $\operatorname{codim} V < \infty\}$. M. González and A. Martinón proved that the quantities B and Γ_u are not equivalent and also that the quantities Γ_u and Γ are not equivalent [5, Theorem 2.7 and Corollary 3.5].

In the terminology of [10], by [3, Theorem 6] it results that u/Γ_u is a perturbation function for $\Phi_+(X,Y)$. Since $\Gamma_u \geq B$ it follows that the perturbation function u/Γ_u is better than the perturbation function u/B. Thus, Theorem 2.14 in [9] is a consequence of the fact that u/Γ_u is a perturbation function for $\Phi_+(X,Y)$.

The above fact can be established from the inequality [7, Proposicion 25.8.4]

$$\Gamma_u(T+S) \leq u(T) + \Gamma(S),\ T, S \in B(X,Y).$$

Indeed, let $u(S) < \Gamma_u(T)$. Then

$$\Gamma_u(T) = \Gamma_u(T+S+(-S)) \leq \Gamma(T+S) + u(S) < \Gamma(T+S) + \Gamma_u(T),$$

and

$$\Gamma(T+S) > 0 \Longrightarrow T+S \in \Phi_+(X,Y).$$

Now we introduce and investigate a new function connected with the lower semi-Fredholm operators.

Definition 2.7. A lower perturbation function is a function δ, which assigns to each pair of complex Banach spaces X, Y, and $T \in B(X,Y)$ a nonnegative number $\delta(T)$, with the following properties:

(2.7.1) $\qquad \delta(\lambda T) = |\lambda| \delta(T),$

(2.7.2) $\qquad \delta(T+K) = \delta(T),\ K \in K(X,Y),$

(2.7.3) $\qquad \delta(T) \leq \|T\|,$

(2.7.4) $\qquad n(T) \leq \delta(T),$

(2.7.5) $\qquad U \subset V \Longrightarrow \delta(Q_U T) \geq \delta(Q_V T),$

where U, V denote closed infinite codimensional subspaces of Y.

We shall give several examples.

Example 2.8. The quantity $\|\cdot\|_\mu$ is a lower perturbation function.

We shall prove only (2.7.4) and (2.7.5). Recall that $\|T\|_\mu = \|T^*\|_q$ [1, Teorema 2.5.2]. Now from [3, Example 3] it follows:

$$\|T\|_\mu = \|T^*\|_q \geq m(T^*) = n(T).$$

Let U and V denote closed infinite codimensional subspaces of Y and $U \subset V$. Let be $\phi : Y/U \to Y/V$ a map defined by $\phi(y+U) = y+V$. Then, by [6, Lemma 3.2] we have

$$\|Q_V T\|_\mu = \|\phi Q_U T\|_\mu \leq \|\phi\|_\mu \|Q_U T\|_\mu \leq \|\phi\| \|Q_U T\|_\mu \leq \|Q_U T\|_\mu.$$

Example 2.9. The quantity $\|\cdot\|_q$ is a lower perturbation function.

We shall prove only (2.7.4) and (2.7.5). Recall that

$$\|T\|_q \geq \|T^*\|_\mu$$

(see [4, Theorem 1 (ii), Proposition 6 (ii)] or [1, Corollary 2.5.4]). Since $\|T^*\|_\mu \geq m(T^*) = n(T)$, we get

$$\|T\|_q \geq n(T).$$

Analogously as in Example 2.8, it can be proved that $\|\cdot\|_q$ has the property (2.7.5).

Example 2.10. The quantity $\|\cdot\|_C$ is a lower perturbation function.

Indeed, since $\|T\|_C \geq \|T\|_q$, we get $\|T\|_C \geq n(T)$. Since $B(Y,Z)K(X,Y) \subset K(X,Z)$, it follows that $\|BA\|_C \leq \|B\|\|A\|_C$, $A \in B(X,Y)$, $B \in B(Y,Z)$. In an analogous way as in Example 2.8, we obtain (2.7.5).

Example 2.11. The quantity ∇ is a lower perturbation function.

Since ∇ is a seminorm on $B(X,Y)$ which annihilates precisely on the set $CS(X,Y)$ and $K(X,Y) \subset CS(X,Y)$ we obtain properties (2.7.1) and (2.7.2). (2.7.3) is obvious.

To prove the property (2.7.4), suppose that V is a closed subspace of Y with $\operatorname{codim} V \geq 1$. Then V^o is a subspace of Y^*, $\dim V^o = \operatorname{codim} V \geq 1$ and $\|Q_V T\| = \|T^* J_{V^o}\|$. Thus

$$\begin{aligned}&\{\|Q_V T\| : V \text{ closed subspace of } Y,\ \operatorname{codim} V \geq 1\}\\ &\qquad\subset \{\|T^* J_M\| : M \text{ subspace of } Y^*,\ \dim M \geq 1\}.\end{aligned}$$

Hence

$$\begin{aligned}\nabla(T) \geq K(T) &= \inf\{\|Q_V T\| : V \text{ closed subspace of } Y,\ \operatorname{codim} V = \infty\}\\ &\geq \inf\{\|Q_V T\| : V \text{ closed subspace of } Y,\ \operatorname{codim} V \geq 1\}\\ &\geq \inf\{\|T^* J_M\| : M \text{ subspace of } Y^*,\ \dim M \geq 1\}\\ &= \inf\{\|T^* J_M\| : M \text{ subspace of } Y^*,\ \dim M = 1\}\\ &= \inf\{\|T^* f\| : f \in Y^*,\ \|f\| = 1\} = m(T^*) = n(T).\end{aligned}$$

Let us remark that

$$n(T) \le K(T) \le \nabla(T) \le \|T\|_q.$$

Since $\nabla(BA) \le \nabla(B)\nabla(A)$, $A \in B(X,Y)$, $B \in B(Y,Z)$, in an analogous way as in Example 2.8, we obtain the property (2.7.5).

Example 2.12. For $T \in B(X,Y)$, set

$$\|T\|_{CS} = \inf\{\|T + C\| : C \in CS(X,Y)\}.$$

The quantity $\|\cdot\|_{CS}$ is a lower perturbation function.

Really, as in the previous example, we obtain that $\|\cdot\|_{CS}$ has properties (2.7.1)-(2.7.3). It is easy to see that $\nabla(T) \le \|T\|_{CS}$ and since $n(T) \le \nabla(T)$, we get $n(T) \le \|T\|_{CS}$. Since $B(Y,Z)CS(X,Y) \subset CS(X,Z)$, it follows that $\|BA\|_{CS} \le \|B\|\|A\|_{CS}$, $A \in B(X,Y)$, $B \in B(Y,Z)$. Now, in an analogous way as in Example 2.8, we obtain the property (2.7.5).

Problem: The quantity v has properties (2.7.1)-(2.7.4), but we do not know whether it has property (2.7.5).

From Definition 2.7 we have

Lemma 2.13. *If δ is a lower perturbation function, then:*

(2.13.1) $\qquad \delta(K) = 0, \ K \in K(X,Y),$

(2.13.2) $\qquad \delta(Q_V) = 1,$

$\qquad$ *V closed infinite codimensional subspace of Y.*

Proof. (2.13.2) follows from the following inequalities:

$$1 = m(J_{V^\circ}) = n(Q_V) \le \delta(Q_V) \le \|Q_V\| = 1. \quad \square$$

In the following δ is a lower perturbation function.

Lemma 2.14. *Let $P \in B(X)$. If $\delta(P) < 1$, then $I + P \in \Phi(X)$ and $i(I+P) = 0$.*

Proof. Assume that $I + P \notin \Phi_-(X)$. By [6, Lemma 5.4] there exists $K \in K(X)$ such that $\operatorname{codim}\overline{R(I+P-K)} = \infty$. Set $U = \overline{R(I+P-K)}$. From $Q_U(I+P-K) = 0$ we get $Q_U = Q_U(K-P)$. Hence, by (2.13.2), (2.7.2) and (2.7.5), it follows that

$$1 = \delta(Q_U) = \delta(Q_U(K-P)) = \delta(Q_U P) \le \delta(P).$$

This contradicts the hypothesis. Thus, $I + P \in \Phi_-(X)$.

Let $0 \le \lambda \le 1$. Then $\delta(\lambda P) < 1$ and therefore $I + \lambda P \in \Phi_-(X)$. Since the index is locally constant it follows that $i(I+P) = i(I) = 0$. Consequently, $I + P \in \Phi(X)$. $\square$

The following proposition can be proved (see [3, Theorem 5]).

Proposition 2.15. $r_e(T) = \lim_{n\to\infty} (\delta(T^n))^{\frac{1}{n}}, \quad T \in B(X).$

Definition 2.16. For $T \in B(X,Y)$, set

$$K_{\delta,V}(T) = \inf\{\delta(Q_W T) : W \supset V\}, \quad K_\delta(T) = K_{\delta,\{0\}}(T),$$
$$\nabla_{\delta,V}(T) = \sup\{K_{\delta,W}(T) : W \supset V\}, \quad \nabla_\delta(T) = \nabla_{\delta,\{0\}}(T),$$

where V, W denote closed infinite codimensional subspaces of Y.

Theorem 2.17. *Let S, $T \in B(X,Y)$. If $\nabla_\delta(T) < K_\delta(T)$, then T, $T+S \in \Phi_-(X,Y)$ and $i(T+S) = i(T)$.*

Proof. Suppose that $T + S \notin \Phi_-(X,Y)$. Then, by [6, Lemma 5.4], there is $K \in K(X,Y)$ such that $\operatorname{codim}\overline{R(T+S-K)} = \infty$. Set $U = \overline{R(T+S-K)}$. Let V be a closed infinite codimensional subspace of Y such that $V \supset U$. Then $Q_V(T+S-K) = 0$, and $Q_V T = Q_V(K-S)$. Therefore

$$\begin{aligned}K_\delta(T) \le K_{\delta,U}(T) &= \inf\{\delta(Q_V T) : V \supset U\} = \inf\{\delta(Q_V(K-S)) : V \supset U\} \\ &= \inf\{\delta(Q_V S) : V \supset U\} = K_{\delta,U}(S) \le \nabla_\delta(S).\end{aligned}$$

This contradicts the hypothesis.

Let $0 \le \lambda \le 1$. Then $\nabla_\delta(\lambda S) < K_\delta(T)$. It follows that $T + \lambda S \in \Phi_-(X,Y)$. Thus T, $T+S \in \Phi_-(X,Y)$ and, since the index is locally constant, we get $i(T) = i(T+S)$. $\square$

Theorem 2.18. *Let $T \in B(X,Y)$. Then $T \in \Phi_-(X,Y) \Longleftrightarrow K_\delta(T) > 0$.*

Proof. Let $K_\delta(T) > 0$. If $S = 0$ then $\nabla_\delta(S) = 0 < K_\delta(T)$, and by Theorem 2.17 we get $T \in \Phi_-(X,Y)$.

Let $T \in \Phi_-(X,Y)$. Then $\operatorname{codim} R(T) < \infty$, and we can express Y as a direct sum $Y = R(T) \oplus V$ where V is a subspace of Y with $\dim V < \infty$. This implies that $Q_V T$ is surjective, i.e. $n(Q_V T) > 0$. Let W be a closed infinite codimensional subspace of Y. Clearly, $\operatorname{codim}(V+W) = \infty$. Now, by (2.7.4) and (2.7.5), we have:

$$n(Q_V T) \le n(Q_{V+W}T) \le \delta(Q_{V+W}T) \le \delta(Q_W T).$$

Consequently,

$$\begin{aligned}K_\delta(T) &= \inf\{\delta(Q_W T) : W \text{ closed subspace of } Y,\ \operatorname{codim} W = \infty\} \\ &\ge n(Q_V T) > 0. \quad \square\end{aligned}$$

Proposition 2.19. *$T \in B(X,Y)$ is strictly cosingular if and only if* $\nabla_\delta(T) = 0$.

Proof. First we shall prove the following inequality

$$v(T) \leq \nabla_\delta(T) \leq \nabla(T). \tag{2.19.1}$$

The right side of the above inequality follows from (2.7.3) and the definitions of ∇ and ∇_δ. To prove the left side, let V be a closed infinite codimensional subspace of Y. We have

$$\delta(Q_W T) \geq n(Q_W T) \geq n(Q_V T),$$

for each closed infinite codimensional subspace W, such that $W \supset V$. It implies

$$K_{\delta,V}(T) = \inf_{W \supset V} \delta(Q_W T) \geq n(Q_V T).$$

Hence

$$\nabla_\delta(T) = \sup_V K_{\delta,V}(T) \geq \sup_V n(Q_V T) = v(T).$$

Now the assertion of Proposition follows from (2.19.1), (1.2) and (1.4). $\square$

Let us remark that if δ is a submultiplicative seminorm then it can be proved that ∇_δ is a submultiplicative seminorm and

$$K_\delta(T+S) \leq K_\delta(T) + \nabla_\delta(S), \quad T,\ S \in B(X,Y).$$

Also in this case we can show that ∇_δ is a lower perturbation function with $\nabla_\delta(T) \leq \delta(T)$ (the property (2.7.4) follows from the inequality $n(T) \leq v(T) \leq \nabla_\delta(T)$ and the property (2.7.5) can be proved analogously as in Example 2.8). Since

$$M(T) \leq K_\delta(T) \leq K(T),$$

where $M(T) = \sup\{n(Q_V T) : \dim V < \infty\}$ [13], from [13, Theorem 8.1] it follows:

$$s_-(T) = \lim_{n \to \infty} (K_\delta(T^n))^{\frac{1}{n}},$$

where $s_-(T) = \inf\{|\lambda| : \lambda I - T \notin \Phi_-(X)\}$.

The next lemma implies that $\nabla(\nabla_\delta) = \nabla_\delta$.

Lemma 2.20. $K_{\delta,V}(T) = \inf\{\nabla_{\delta,W}(T) : W \supset V\}$, *where V, W denote closed infinite codimensional subspaces of Y.*

Proof. Since

$$\nabla_{\delta,W}(T) \geq K_{\delta,W}(T) \geq K_{\delta,V}(T),$$

for each W with $W \supset V$, we get

$$\inf\{\nabla_{\delta,W}(T) : W \supset V\} \geq K_{\delta,V}(T).$$

In the following M, N are closed infinite codimensional subspace of Y. Let $M \supset W$. From (2.7.5) it follows that

$$K_{\delta,M}(T) = \inf_{N \supset M} \delta(Q_N T) \leq \delta(Q_M T) \leq \delta(Q_W T).$$

Consequently

$$\nabla_{\delta,W}(T) = \sup_{M \supset W} K_{\delta,M}(T) \leq \delta(Q_W T).$$

This implies

$$\inf_{W \supset V} \nabla_{\delta,W}(T) \leq \inf_{W \supset V} \delta(Q_W T) = K_{\delta,V}(T). \quad \square$$

If $\delta = \|\cdot\|_q$ we shall write K_q (∇_q) instead of K_δ (∇_δ). Analogously, we introduce K_μ (∇_μ), K_C (∇_C), K_∇ (∇_∇) i K_{CS} (∇_{CS}). From Lemma 2.20 it follows that $K_\nabla = K$ and $\nabla_\nabla = \nabla$.

In an analogous way as Lemma 2.4 the next lemma can be proved.

Lemma 2.21. *If there exists a constant $c > 0$ such that $\delta(T) \geq c\nabla(T)$, for each $T \in B(X,Y)$, then*

$$\begin{aligned} K_\delta(T) &\leq K(T) \leq \frac{1}{c} K_\delta(T), \\ \nabla_\delta(T) &\leq \nabla(T) \leq \frac{1}{c} \nabla_\delta(T), \ T \in B(X,Y). \end{aligned}$$

Recall that

$$\begin{aligned} &\|T\|_q \geq \nabla(T), \\ &\|T\|_\mu \geq \frac{1}{2}\|T\|_q \geq \frac{1}{2}\nabla(T), \\ &\|T\|_C \geq \|T\|_q \geq \nabla(T), \\ &\|T\|_{CS} \geq \nabla(T), \quad T \in B(X,Y). \end{aligned}$$

Now, by Lemma 2.21 we obtain

$$\begin{aligned} &K_q(T) = K(T), \quad \nabla_q(T) = \nabla(T), \\ &K_\mu(T) \leq K(T) \leq 2K_\mu(T), \ \nabla_\mu(T) \leq \nabla(T) \leq 2\nabla_\mu(T), \\ &K_C(T) = K(T), \quad \nabla_C(T) = \nabla(T), \\ &K_{CS}(T) = K(T), \quad \nabla_{CS}(T) = \nabla(T). \end{aligned} \tag{2.22}$$

The equalities in (2.22) were proved in [11, Summary and discussion, Remark 2]. However, our proof is different from this one.

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