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EXISTENCE OF OSCILLATORY AND NONOSCILLATORY SOLUTIONS FOR A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. We shall establish sufficient conditions for the existence of at least one nonoscillatory solution and for oscillation of all solution of the following system of differential equations

$$u_i' = |u_{3-i}|^{\lambda_i} \mathrm{sgn} u_{3-i} + (-1)^{i-1} b_i(t) u_i \quad (i = 1, 2), \quad \lambda_1 \lambda_2 \neq 1.$$

Introduction

The following system of differential equations

$$\text{(EF)} \qquad v_1' = c_1(s)|v_2|^{\lambda_1} \mathrm{sgn}\, v_2, \qquad v_2' = c_2(s)|v_1|^{\lambda_2} \mathrm{sgn}\, v_1$$

is known in the literature as the system od the Emden–Fowler type. The oscillatory properties of its solutions have been studied, during the last twenty years, by many authors ([1],[4]–[11]). A survey on such results and fairly extensive bibliography of the earlier work can be found in the book of Mirzov D.D. [7].

The following change of variables

$$t = \int_0^s c_2(\tau) \left(\frac{c_2(\tau)}{c_1(\tau)} \right)^{-\frac{1+\lambda_2}{2+\lambda_1+\lambda_2}} d\tau,$$

$$v_i(s) = \left(\frac{c_i(s)}{c_{3-i}(s)} \right)^{\frac{1}{2+\lambda_1+\lambda_2}} u_i(t) \quad (i = 1, 2)$$

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reduces the Emden–Fowler system to the following nonlinear system of differential equations

$$u_1' = |u_2|^{\lambda_1}\mathrm{sgn}\, u_2 + b(t)u_1, \qquad u_2' = |u_1|^{\lambda_2}\mathrm{sgn}\, u_1 - b(t)u_2$$

where

$$b(t) = -\frac{1}{2+\lambda_1+\lambda_2}\left(\frac{c_1(s)}{c_2(s)}\right)^{-1-\frac{1+\lambda_2}{2+\lambda_1+\lambda_2}}\frac{1}{c_2(s)}\left(\frac{c_1(s)}{c_2(s)}\right)'.$$

We shall consider the nonlinear system of differential equations

$$\begin{aligned} u_1' &= |u_2|^{\lambda_1}\mathrm{sgn}\, u_2 + b_1(t)u_1, \\ u_2' &= |u_1|^{\lambda_2}\mathrm{sgn}\, u_1 - b_2(t)u_2 \end{aligned} \tag{1}$$

where the functions b_i $(i=1,2)$ are nonnegative and summable on each finite segment of the interval $[0,+\infty)$ and

$$\lambda_i > 0 \ (i=1,2), \qquad \lambda_1\lambda_2 \neq 1.$$

A solution $(u_1(t), u_2(t))$ of the system (1) which is defined on some positive halfline $[t_0,+\infty)$, where $t_0 \geq 0$ depends on the particular solution, is called *proper* if

$$\sup\{|u_1(\tau)| + |u_2(\tau)| : t \leq \tau < +\infty\} > 0 \quad \text{for all} \quad t \in [t_0,+\infty).$$

A proper solution $(u_1(t), u_2(t))$ of the system (1) is called *oscillatory (weakly oscillatory)* if both components (at least one component) have sequence of zeroes convergent to $+\infty$. If we can find $t_* > t_0$ such that both components (at least one component) are different from zero on $[t_*,+\infty)$, then the proper solution $(u_1(t), u_2(t))$ is said to be *nonoscillatory (weakly nonoscillatory)*.

Denote the set of functions which are summable on each finite segment of the interval $[0,+\infty)$ by $L_{loc}([0,+\infty))$.

1. Nonoscillation theorems

Let the functions $a_i \in L_{loc}([0,+\infty)))$ satisfy the following conditions

$$a_i(t) \geq (-1)^i \quad (i=1,2) \quad \text{for} \quad t \geq 0 \tag{1.1}$$

and let the functions b_i in the halfspace

$$D = \{(t,u_1,u_2) : t \geq 0, -\infty < u_1, u_2 < +\infty\}$$

satisfy the conditions

$$
\begin{aligned}
&a_1(t)|u_2|^{\lambda_1} \le b_1(t)u_1 \operatorname{sgn} u_2 \le M a_1(t)|u_2|^{\lambda_1},\\
&a_2(t)|u_1|^{\lambda_2} \le b_2(t)u_2 \operatorname{sgn} u_1 \le M a_2(t)|u_1|^{\lambda_2}
\end{aligned}
\tag{1.2}
$$

where $M = \text{const.} \ge 1$.

Under appropriate assumptions, using the previous inequalities, we shall prove that the system (1) can be reduced to the Emden-Fowler system of differential inequations. Then, following Mirzov's methods ([1], [5]) for the Emden–Fowler system we shall prove the nonoscillation theorems.

We shall need the following two lemmas.

Lemma 1.1. *For every $t_0 \in [0, +\infty)$ the trivial solution is the only solution which satisfies the initial condition*

$$u_1(t_0) = u_2(t_0) = 0.$$

Proof. Because of the assumed relations (1.1) and (1.2) we have

$$u_1'(t) \operatorname{sgn} u_2(t) \ge (1 + a_1(t))|u_2(t)|^{\lambda_1} \ge 0, \tag{1.3}$$

$$u_2'(t) \operatorname{sgn} u_1(t) \le (1 - a_2(t))|u_1(t)|^{\lambda_2} \le 0. \tag{1.4}$$

Suppose the contrary. Then we can find an $\varepsilon > 0$ such that

$$u_1(t) \ne 0 \quad \text{for} \quad t \in (t_0, t_0 + \varepsilon).$$

We distinguish two cases.

Case 1. $u_1(t) > 0$ for $t \in (t_0, t_0 + \varepsilon)$. From (1.4) we deduce that $u_2(t)$ is the nonincreasing function. Hence, $u_2(t) \le u_2(t_0) = 0$ for $t > t_0$. Now, we conclude because of (1.3) that $u_1(t)$ is the nonincreasing function, which implies that for all $t > t_0$ is $u_1(t) \le u_1(t_0) = 0$. The obtained contradiction proves that Case 1. is impossible.

Case 2. $u_1(t) < 0$ for $t \in (t_0, t_0 + \varepsilon)$. From (1.4) we obtain that $u_2(t) \ge u_2(t_0) = 0$ for $t > t_0$, which implies, according to (1.3), that $u_1(t)$ is the nondecreasing function. Consequently, $u_1(t) \ge u_1(t_0) = 0$ for $t > t_0$. Thus, Case 2. is also impossible and the lemma is proved.

Lemma 1.2. *An arbitrary weakly nonoscillatory (weakly oscillatory) solution of the system* (1) *is nonoscillatory (oscillatory).*

Proof. Suppose that the system (1) has a nontrivial weakly nonoscillatory solution $(u_1(t), u_2(t))$. Then, we can find $t_* > t_0$ such that $u_2(t) \ne 0$ for all

$t \geq t_*$. If we assume the contrary, $u_1(t)$ has a sequence of zeroes convergent to $+\infty$. Using (1), (1.1) and (1.2), it follows from

$$\begin{aligned} u_1'(t)\operatorname{sgn} u_2(t_*) &= |u_2(t)|^{\lambda_1} + b_1(t)u_1(t)\operatorname{sgn} u_2(t) \\ &\geq (1+a_1(t))|u_2(t)|^{\lambda_1} \geq 0 \quad \text{for} \quad t \geq t_* \end{aligned}$$

that u_1 is monotone on $[t_*, +\infty)$ and we can find $t^* \geq t_*$ such that $u_1(t) = 0$ for $t \geq t^*$. As we have just got a contradiction, the lemma is proved i.e. $(u_1(t), u_2(t))$ is nonoscillatory.

In the following two theorems, corresponding to the cases $\lambda_1\lambda_2 < 1$ and $\lambda_1\lambda_2 > 1$, sufficient conditions for the existence of at least one nonoscillatory solution of the system (1) will be given. These theorems are extensions of oscillation theorems (Theorem 11.3., Theorem 11.4. in [7]) for the Emden–Fowler system.

Theorem 1.1. *Let $\lambda_1\lambda_2 < 1$ and for some $i \in \{1,2\}$ the condition*

$$\int_0^{+\infty} \left(M^{i-1}|a_{3-i}(t)| + i - 1\right) \left[\int_0^t \left(M^{2-i}|a_i(s)| + 2 - i\right) ds\right]^{\lambda_{3-i}} dt < +\infty,$$

be satisfied. Then, the system (1) *has at least one nonoscillatory solution.*

Proof. We shall take $i = 1$ since the case when $i = 2$ can be considered in a similar way. Let $t_0 > 0$ be such that

$$\int_{t_0}^{+\infty} a_2(t) \left(\int_{t_0}^t (M|a_1(s)| + 1)\, ds\right)^{\lambda_2} dt = K < +\infty \tag{1.5}$$

We shall prove that the solution $(u_1(t), u_2(t))$ of the system (1) which satisfies the initial conditions

$$u_1(t_0) = 0, \quad |u_2(t_0)|^{1-\lambda_1\lambda_2} > KM.$$

is nonoscillatory. According to Lemma 1.2., it is enough to prove that $|u_1(t)| > 0$ for all $t > t_0$.

Suppose the contrary that there exist sequences $\{t_{in}\}_{n=1}^{+\infty}$ $(i = 1,2)$, $t_{11} = t_0$ such that $t_{2n} > t_{1n} > n$ $(n = 1,2,\ldots)$,

$$u_1(t_{in}) = 0 \quad (i = 1,2) \quad \text{and} \quad u_1(t) \neq 0 \quad \text{for} \quad t_{1n} < t < t_{2n}. \tag{1.6}$$

According to Lemma 1.1., $u_2(t_{in}) \neq 0$, $(i = 1,2,\ n = 1,2,\ldots)$. In the first place, we shall prove that there exist the points $\tau_n \in (t_{1n}, t_{2n})$ such that $u_2(\tau_n) = 0$. Suppose that $u_2(t) \neq 0$ for every $t \in (t_{1n}, t_{2n})$. Then

$$u_1'(t)\operatorname{sgn} u_2(t_{1n}) \geq (1+a_1(t))|u_2(t)|^{\lambda_1} \geq 0 \quad \text{for} \quad t \in (t_{1n}, t_{2n}),$$

which means that $u_1(t)$ is monotone on (t_{1n}, t_{2n}). Therefore, for $t \in (t_{1n}, t_{2n})$ we have

$$0 = u_1(t_{in}) \le u_1(t) \le u_1(t_{jn}) = 0 \quad \text{for} \quad (i, j = 1, 2)\; i \ne j.$$

The contradiction which we have just obtained proves the existence of $\tau_n \in (t_{1n}, t_{2n})$ such that $u_2(\tau_n) = 0$ and let τ_n be the first such point.

In that case $u_1(t)u_2(t) \ne 0$ for $t \in (t_{1n}, \tau_n)$. We shall next prove that

$$u_1(t)u_2(t) > 0 \quad \text{for} \quad t \in (t_{1n}, \tau_n). \tag{1.7}$$

If this is not true, using (1.1) and (1.2), we obtain

$$-|u_1(t)|' \ge (1 + a_1(t))|u_2(t)|^{\lambda_1} \ge 0 \quad \text{for} \quad t \in (t_{1n}, \tau_n),$$

which leads us to the contradiction, since we have then that $|u_1(t)| \le |u_1(t_{1n})| = 0$ for $t > t_{1n}$. Thus, the validity of (1.7) is proved.

By (1.1), (1.2) and (1.7) the following inequalities are valid for $t \in [t_0, \tau_1]$

$$0 \le |u_1(t)|' \le (1 + M|a_1(t)|)|u_2(t)|^{\lambda_1} \tag{1.8}$$

$$0 \ge |u_2(t)|' \ge (1 - Ma_2(t))|u_1(t)|^{\lambda_2} \ge -Ma_2(t)|u_1(t)|^{\lambda_2}. \tag{1.9}$$

Integrating (1.8) over $[t_0, t]$, we obtain

$$\begin{aligned} |u_1(t)| &\le |u_1(t_0)| + \int_{t_0}^{t} (1 + M|a_1(s)|)|u_2(s)|^{\lambda_1}\, ds \\ &\le |u_2(t_0)|^{\lambda_1} \int_{t_0}^{t} (1 + M|a_1(s)|)\, ds \quad \text{for} \quad t \in [t_0, \tau_1]. \end{aligned}$$

Consequently, according to (1.5) and (1.9), we have

$$\begin{aligned} |u_2(t)| &\ge |u_2(t_0)| - M|u_2(t_0)|^{\lambda_1\lambda_2} \int_{t_0}^{t} a_2(s) \left(\int_{t_0}^{s} (1 + M|a_1(\tau)|)\, d\tau \right)^{\lambda_2} ds \\ &\ge |u_2(t_0)| - M|u_2(t_0)|^{\lambda_1\lambda_2} K > 0 \quad \text{for} \quad t \in [t_0, \tau_1], \end{aligned}$$

which assures that $|u_2(\tau_1)| > 0$. This contradicts the choice of the point τ_1. The proof is therefore completed.

Theorem 1.2. *Let $\lambda_1\lambda_2 > 1$ and for some $i \in \{1,2\}$ the following conditions*

$$\int_0^{+\infty} \left(M^{i-1}|a_{3-i}(t)| + i - 1\right) dt < +\infty \tag{1.10-i}$$

and

(1.11-i)

$$\int_0^{+\infty} \left(M^{2-i}|a_i(t)| + 2 - i\right) \left[\int_t^{+\infty} \left(M^{i-1}|a_{3-i}(s)| + i - 1\right) ds\right]^{\lambda_i} dt < +\infty.$$

be satisfied. Then, there exists a nonoscillatory solution of the system (1).

Proof. In the first place, we suppose that conditions (1.10-1) and (1.11-1) are satisfied. We choose $t_0 \geq 0$ such that

$$M \int_{t_0}^{+\infty} a_2(t)\, dt < 1 \tag{1.12}$$

$$M^{\lambda_1} \int_{t_0}^{+\infty} (1 + M|a_1(t)|) \left(\int_t^{+\infty} a_2(s)\, ds\right)^{\lambda_1} dt < \frac{1}{2}. \tag{1.13}$$

For arbitrary $n \in N$ we consider the solution $(u_{n1}(t), u_{n2}(t))$ of the system (1) which satisfies the initial conditions

$$u_{n1}(t_0 + n) = 1, \quad u_{n2}(t_0 + n) = 0.$$

We shall prove that this solution is defined on $[t_0, t_0 + n]$ and satisfies the inequalities

$$\frac{1}{2} < u_{n1}(t) \leq 1, \quad 0 \leq u_{n2}(t) < 1 \tag{1.14}$$

on this segment. Since there exists an $\varepsilon > 0$ such that $u_{n1}(t) > 0$ for all $t \in (t_0 + n - \varepsilon, t_0 + n)$, using (1), (1.1) and (1.2), we get

$$u'_{n2}(t) = |u_{n1}(t)|^{\lambda_2} - b_2(t)u_{n2}(t) \leq (1 - a_2(t))|u_{n1}(t)|^{\lambda_2} \leq 0,$$

for $t \in (t_0 + n - \varepsilon, t_0 + n)$. Accordingly, for all $t \in (t_0 + n - \varepsilon, t_0 + n)$ is $u_{n2}(t) \geq u_{n2}(t_0 + n) = 0$. Then, from (1), (1.1) and (1.2), we obtain for $t \in (t_0 + n - \varepsilon, t_0 + n)$

$$u'_{n1}(t) = |u_{n2}(t)|^{\lambda_1} + b_1(t)u_{n1}(t) \geq (1 + a_1(t))|u_{n2}(t)|^{\lambda_1} \geq 0.$$

Therefore, for $t \in (t_0 + n - \varepsilon, t_0 + n)$ is $u_{n1}(t) \leq u_{n1}(t_0 + n) = 1$. If the assertion (1.14) is false there exists $t_* \in [t_0, t_0 + n)$ such that the inequalities (1.14) are valid on $(t_*, t_0 + n)$ and

$$u_{n1}(t_*) = \frac{1}{2} \tag{1.15}$$

or

$$u_{n2}(t_*) = 1. \tag{1.16}$$

Then, by (1.2), for $t \in (t_*, t_0 + n)$ we have

$$u'_{n1}(t) \leq (1 + M|a_1(t)|)|u_{n2}(t)|^{\lambda_1}, \tag{1.17}$$

$$u'_{n2}(t) \geq (1 - Ma_2(t))|u_{n1}(t)|^{\lambda_2} > -Ma_2(t). \tag{1.18}$$

Integrating (1.18) from t_* to $t_0 + n$, according to (1.12), we get

$$u_{n2}(t_*) < M \int_{t_*}^{t_0+n} a_2(s)\, ds < 1,$$

which is contradictory to (1.16). The relation (1.17) implies that

$$u'_{n1}(t) < (1 + M|a_1(t)|)M^{\lambda_1} \left(\int_t^{+\infty} a_2(\tau)\, d\tau \right)^{\lambda_1} \quad \text{for} \quad t \in (t_*, t_0 + n).$$

Integrating the previous inequality from t_* to $t_0 + n$ we obtain the estimate

$$1 - u_{n1}(t_*) < M^{\lambda_1} \int_{t_*}^{t_0+n} (1 + M|a_1(t)|) \left(\int_t^{+\infty} a_2(\tau)\, d\tau \right)^{\lambda_1} dt,$$

which together with (1.13) leads us to the contradiction. Consequently, the solution $(u_{n1}(t), u_{n2}(t))$ is defined on $[t_0, t_0 + n]$ and on this segment satisfies the conditions (1.14).

The functions $v_{ni}(t)$ defined by the following

$$v_{ni}(t) = \begin{cases} u_{ni}(t) & , \ t_0 \leq t \leq t_0 + n \\ u_{ni}(t_0 + n) & , \ t \geq t_0 + n \end{cases} \tag{1.19}$$

according to (1.14), satisfy the inequalities

$$\frac{1}{2} < v_{n1}(t) \leq 1, \quad 0 \leq v_{n2}(t) < 1 \quad (n = 1, 2, \ldots) \quad \text{for} \quad t \geq t_0,$$

and according to (1), (1.1) and (1.2) the inequalities

$$v'_{n1}(t) \le (1 + M|a_1(t)|)|v_{n2}(t)|^{\lambda_1} < 1 + M|a_1(t)|,$$
$$v'_{n2}(t) \le (1 - a_2(t))|v_{n1}(t)|^{\lambda_2} < \frac{1 - a_2(t)}{2^{\lambda_2}}.$$

The sequence $\{(v_{n1}(t), v_{n2}(t))\}_{n=1}^{+\infty}$ is equicontinuous and equibounded on each finite segment of the interval $[t_0, +\infty)$. According to the Theorem Ascoli-Arzela ([3], p.4) this sequence tends uniformly to $(u_1(t), u_2(t))$ as $n \to +\infty$ on each finite segment of the interval $[t_0, +\infty)$, provided that

$$\frac{1}{2} < u_1(t) \le 1, \quad 0 \le u_2(t) < 1 \quad \text{for} \quad t \ge t_0. \tag{1.20}$$

We shall prove that $(u_1(t), u_2(t))$ is the solution of the system (1) on $[t_0, +\infty)$. For an arbitrary segment $[t_0, T]$, we choose $n \in N$ sufficiently large that $(v_{n1}(t), v_{n2}(t))$ is a solution of the system

$$v_{n1}(t) = v_{n1}(t_0) + \int_{t_0}^{t} \Big[|v_{n2}(s)|^{\lambda_1} \operatorname{sgn} v_{n2}(s) + b_1(s) v_{n1}(s) \Big] \, ds$$
$$v_{n2}(t) = v_{n2}(t_0) + \int_{t_0}^{t} \Big[|v_{n1}(s)|^{\lambda_2} \operatorname{sgn} v_{n1}(s) - b_2(s) v_{n2}(s) \Big] \, ds$$

on $[t_0, T]$. Using the Lebeg's theorem, we conclude that $(u_1(t), u_2(t))$ is the solution of the system (1) on $[t_0, T]$. The observed segment $[t_0, T]$ is arbitrary and for this reason $(u_1(t), u_2(t))$ is the solution of the system (1) on $[t_0, +\infty)$.

On the other hand, according to Lemma 1.2. and (1.20), this solution is nonoscillatory.

Now, we suppose that conditions (1.10-2) and (1.11-2) are satisfied. We choose $t_0 \ge 0$ such that

$$\int_{t_0}^{+\infty} (1 + M|a_1(t)|) \, dt < 1 \tag{1.21}$$

$$M \int_{t_0}^{+\infty} a_2(t) \left(\int_{t}^{+\infty} (1 + M|a_1(s)|) \, ds \right)^{\lambda_2} dt < \frac{1}{2}. \tag{1.22}$$

For arbitrary $n \in N$ we consider the solution $(u_{n1}(t), u_{n2}(t))$ of the system (1) which satisfies the initial conditions

$$u_{n1}(t_0 + n) = 0, \quad u_{n2}(t_0 + n) = 1.$$

We shall prove that this solution is defined on $[t_0, t_0 + n]$ and satisfies the inequalities

$$-1 < u_{n1}(t) \le 0, \quad \frac{1}{2} < u_{n2}(t) \le 1. \tag{1.23}$$

Since there exists an $\varepsilon > 0$ such that for $t \in (t_0 + n - \varepsilon, t_0 + n)$, $u_{n2}(t) > 0$, according to (1), (1.1) and (1.2), we get

$$u'_{n1}(t) = |u_{n2}(t)|^{\lambda_1} + b_1(t)u_{n1}(t) \ge (1 + a_1(t))|u_{n2}(t)|^{\lambda_1} \ge 0.$$

Hence, for all $t \in (t_0 + n - \varepsilon, t_0 + n)$ is $u_{n1}(t) \le u_{n1}(t_0 + n) = 0$. Now, from (1), (1.1) and (1.2), we obtain

$$-u'_{n2}(t) = |u_{n1}(t)|^{\lambda_2} + b_2(t)u_{n2}(t) \le (1 - a_2(t))|u_{n1}(t)|^{\lambda_2} \le 0,$$

which implies that for all $t \in (t_0 + n - \varepsilon, t_0 + n)$ is $u_{n2}(t) \le u_{n2}(t_0 + n) = 1$. Suppose that (1.23) is not true. Then there exists $t^* \in [t_0, t_0 + n)$ such that the inequalities (1.23) are valid on $(t^*, t_0 + n)$ and

$$u_{n1}(t^*) = -1 \tag{1.24}$$

or

$$u_{n2}(t^*) = \frac{1}{2}. \tag{1.25}$$

Then, by (1.2), for all $t \in (t^*, t_0 + n)$, we obtain the estimates

$$u'_{n1}(t) \le 1 + M|a_1(t)|, \tag{1.26}$$

$$-u'_{n2}(t) \ge -Ma_2(t)|u_{n1}(t)|^{\lambda_2}. \tag{1.27}$$

Integrating (1.26) from t^* to $t_0 + n$, we get

$$u_{n1}(t^*) \ge -\int_{t^*}^{t_0+n} (1 + M|a_1(s)|)\, ds > -1,$$

which contradicts (1.24). Since $u_{n1}(t) \le 0$ on $(t^*, t_0 + n)$, from the previous inequality, we have

$$|u_{n1}(t)|^{\lambda_2} \le \left(\int_t^{+\infty} (1 + M|a_1(s)|)\, ds\right)^{\lambda_2} \quad \text{for} \quad t \in (t^*, t_0 + n),$$

which in view of (1.27), implies that

$$-u_{n2}'(t) \geq -Ma_2(t)\left(\int_t^{+\infty}(1+M|a_1(\tau)|)\,d\tau\right)^{\lambda_2} \quad \text{for} \quad t \in (t^*, t_0+n).$$

Integrating the previous inequality from t^* to t_0+n, we have that

$$-1+u_{n2}(t^*) \geq -M\int_{t^*}^{t_0+n} a_2(t)\left(\int_t^{+\infty}(1+M|a_1(\tau)|)\,d\tau\right)^{\lambda_2} dt.$$

Because of the condition (1.22), we have just got the contradiction. Thus, the solution $(u_{n1}(t), u_{n2}(t))$ is defined on $[t_0, t_0+n]$ and on this segment satisfies the conditions (1.23).

As in the first case, we can prove that the sequence $\{(v_{n1}(t), v_{n2}(t))\}_{n=1}^{+\infty}$ defined by (1.19) tends uniformly to the nonoscillatory solution of the system (1).

2. Oscillation theorems

In this section we shall suppose that the functions $a_i \in L_{loc}([0,+\infty))$ satisfy the following conditions

$$a_1(t) \geq 0, \quad a_2(t) \geq 1 \quad \text{for} \quad t \geq 0. \tag{2.1}$$

It will be assumed that the functions b_i in the halfspace

$$D = \{(t, u_1, u_2) : t \geq 0, -\infty < u_1, u_2 < +\infty\}$$

satisfy the following conditions

$$\begin{aligned} b_1(t)u_1 \operatorname{sgn} u_2 &\geq a_1(t)|u_2|^{\lambda_1}, \\ b_2(t)u_2 \operatorname{sgn} u_1 &\geq a_2(t)|u_1|^{\lambda_2}. \end{aligned} \tag{2.2}$$

Following the approach of integration in the case for the Emden–Fowler system ([4], [5]) we shall prove the oscillation theorems for the system (1).

The following lemma will be needed.

Lemma 2.1. *Let for some $i \in \{1,2\}$ the condition*

$$\int^{+\infty}(a_i(t)-i+1)\,dt = +\infty, \tag{2.3}$$

be fulfilled. Then, an arbitrary nonoscillatory solution $(u_1(t), u_2(t))$ *of the system* (1) *satisfies the condition*

$$(-1)^{i-1} u_1(t) u_2(t) > 0$$

for all large values of t.

Proof. Suppose that the following condition

$$\int_{t_0}^{+\infty} a_1(t)\, dt = +\infty, \tag{2.4}$$

is satisfied and that the system (1) has a nonoscillatory solution $(u_1(t), u_2(t))$ which exists on the ray $[t_0, +\infty)$ and satisfies the following condition

$$u_1(t) u_2(t) < 0 \quad \text{for} \quad t \geq t_0.$$

Then, by (1), (2.1) and (2.2), the following inequalities are valid

$$-|u_1(t)|' \geq (1 + a_1(t)) |u_2(t)|^{\lambda_1} > 0, \tag{2.5}$$

$$-|u_2(t)|' \leq (1 - a_2(t)) |u_1(t)|^{\lambda_2} \leq 0 \tag{2.6}$$

for all $t \geq t_0$. It follows from (2.6) that $|u_2(t)| \geq |u_2(t_0)|$ for all $t \geq t_0$, which implies, in view of (2.5), that $-|u_1(t)|' > a_1(t)|u_2(t_0)|^{\lambda_1}$. Integrating the previous inequality, we get

$$\int_{t_0}^{+\infty} a_1(s)\, ds < |u_1(t_0)| |u_2(t_0)|^{-\lambda_1},$$

which is contradictory to (2.4). By similar arguments we can prove the case when $i = 2$.

We distinguish two cases, when $\lambda_1 \lambda_2 < 1$ and $\lambda_1 \lambda_2 > 1$ and in both cases we shall establish sufficient conditions for oscillation of all solutions of the system (1), which are extensions of the conditions for oscillation of all solutions of the Emden–Fowler system (Theorem 11.3., Theorem 11.4., Theorem 12.9. in [7]).

Theorem 2.1. *Let* $\lambda_1 \lambda_2 < 1$ *and let for some* $i \in \{1, 2\}$ *conditions* (2.3) *and*

$$\int_a^{+\infty} (a_{3-i}(t) + i - 2) \left(\int_a^t (a_i(s) - i + 1)\ ds \right)^{\lambda_{3-i}} dt = +\infty. \tag{2.7}$$

be fulfilled. Then, all nontrivial solutions of the system (1) *are oscillatory.*

Proof. Take $i = 1$. Since an arbitrary weakly oscillatory solution of the system (1) is oscillatory, it is enough to prove that the system (1) does not have any nonoscillatory solution. Suppose that the system (1) has a nonoscillatory solution $(u_1(t), u_2(t))$ which satisfies the conditions (2.3) and (2.7). According to Lemma 2.1. there exists t_0 such that for all $t \geq t_0$ the following inequalities

$$u_1(t)u_2(t) > 0$$

$$|u_1(t)|' > a_1(t)|u_2(t)|^{\lambda_1} \geq 0, \tag{2.8}$$

$$|u_2(t)|' \leq (1 - a_2(t))|u_1(t)|^{\lambda_2} \leq 0 \tag{2.9}$$

are valid. Integrating (2.8) from t_0 to t we get

$$\begin{aligned}|u_1(t)| &> |u_1(t_0)| + \int_{t_0}^{t} a_1(s)|u_2(s)|^{\lambda_1}\,ds \\ &> |u_2(t)|^{\lambda_1}\int_{t_0}^{t} a_1(s)\,ds.\end{aligned}$$

Then, from (2.9) we obtain

$$\int_{t_0}^{t} |u_2(s)|'|u_2(s)|^{-\lambda_1\lambda_2}\,ds < \int_{t_0}^{t}(1 - a_2(s))\left(\int_{t_0}^{s} a_1(\tau)\,d\tau\right)^{\lambda_2} ds.$$

Consequently,

$$\int_{t_0}^{t}(a_2(s) - 1)\left(\int_{t_0}^{s} a_1(\tau)\,d\tau\right)^{\lambda_2} ds < \frac{|u_2(t_0)|^{1-\lambda_1\lambda_2}}{1 - \lambda_1\lambda_2} \quad \text{for} \quad t \geq t_0,$$

which contradicts the initial condition. The case when $i = 2$ can be considered in a similar way.

Theorem 2.2. *Let* $\lambda_1\lambda_2 > 1$ *and let for some* $i \in \{1, 2\}$ *the condition* (2.3) *be fulfilled. If one of the following conditions*

$$\int_0^{+\infty}(a_{3-i}(t) + i - 2)\,dt = +\infty, \tag{2.10}$$

or

$$\begin{aligned}&\int_0^{+\infty}(a_{3-i}(t) + i - 2)\,dt < +\infty, \\ &\int_0^{+\infty}(a_i(t) - i + 1)\left(\int_t^{+\infty}(a_{3-i}(\tau) + i - 2)\,d\tau\right)^{\lambda_i} dt = +\infty.\end{aligned} \tag{2.11}$$

is satisfied, all nontrivial solutions of the system (1) *are oscillatory.*

Proof. Let $i = 1$ and suppose that the system (1) possesses a weakly nonoscillatory solution $(u_1(t), u_2(t))$. According to Lemma 1.2. this solution is nonoscillatory and applying Lemma 2.1. we obtain for $t \geq t_0$ the following inequalities

$$u_1(t)u_2(t) > 0$$

$$|u_1(t)|' > a_1(t)|u_2(t)|^{\lambda_1} \geq 0, \tag{2.12}$$

$$|u_2(t)|' \leq (1 - a_2(t))|u_1(t)|^{\lambda_2} \leq 0. \tag{2.13}$$

Since $|u_1(t)|$ is the increasing function and $1 - a_2(t)$ is the negative function, for all $t \geq t_0$ is $(1 - a_2(t))|u_1(t)|^{\lambda_2} < (1 - a_2(t))|u_1(t_0)|^{\lambda_2}$. Then, if we integrate (2.13) we get

$$0 < |u_2(t)| < |u_2(t_0)| + |u_1(t_0)|^{\lambda_2} \int_{t_0}^{t} (1 - a_2(s))\, ds,$$

i.e.

$$\int_{t_0}^{t} (a_2(s) - 1)\, ds < |u_2(t_0)||u_1(t_0)|^{-\lambda_2} \quad \text{for} \quad t \geq t_0,$$

which contradicts the condition (2.10).

Integrating (2.13) from t to $+\infty$ we get

$$\begin{aligned} |u_2(t)| &\geq \int_{t}^{+\infty} (a_2(s) - 1)|u_1(s)|^{\lambda_2}\, ds \\ &> |u_1(t)|^{\lambda_2} \int_{t}^{+\infty} (a_2(s) - 1)\, ds. \end{aligned}$$

The previous inequality together with (2.12) implies for $t \geq t_0$

$$\int_{t_0}^{t} |u_1(s)|'|u_1(s)|^{-\lambda_1\lambda_2}\, ds > \int_{t_0}^{t} a_1(s) \left(\int_{s}^{+\infty} (a_2(\tau) - 1)\, d\tau \right)^{\lambda_1} ds,$$

i.e.

$$\int_{t_0}^{t} a_1(s) \left(\int_{s}^{+\infty} (a_2(\tau) - 1)\, d\tau \right)^{\lambda_1} ds < \frac{|u_1(t_0)|^{1-\lambda_1\lambda_2}}{\lambda_1\lambda_2 - 1},$$

which contradicts the condition (2.11). The obtained contradictions prove that the system (1) does not have weakly nonoscillatory solution. According to Lemma 1.2. this is sufficient to establish the desired conclusion.

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