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AN ASYMPTOTIC ANALYSIS OF A LINEAR DIFFERENTIAL EQUATION

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We wish to study the asymptotic behavior of the solution of the differential equation

$$u^{(n)} - qu = 0 \tag{1}$$

on $[0,\infty)$, particularly with respect to oscillation. A continuous function f from $[0,\infty)$ to $(-\infty,+\infty)$ is called oscillatory if and only if the set $\{t : t \geq 0,\ f(t) = 0\}$ is unbounded. Let q be a continuous function from $[0,\infty)$ to $(0,+\infty)$, and let n be an integer $n \geq 2$.

It is clear that (1) has nonoscillatory solutions. In particular, if k is an integer in $[1,n]$ and z_k solves

$$z_k(t) = \frac{t^{k-1}}{(k-1)!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} q(s) z_k(s)\, ds \tag{2}$$

on $[0,\infty)$, the z_k is a nonoscillatory solution of (1). We shall call a solution u of (1) strongly increasing if and only if each of $u, u', u'', \dots, u^{(n-1)}$ is eventually positive.

On the other hand, it is known ([1]) that there is a positive solution u of (1) such that $(-1)^k u^{(k)} > 0$ for each k. We shall call a solution u of (1) strongly decreasing if and only if $(-1)^k u^{(k)}$ is eventually positive for each integer $k \in [0, n-1]$. Since we know that there exist strongly increasing and strongly decreasing solutions, the best conclusion one can hope for in an oscillation theorem is that every eventually positive solution is either strongly increasing or strongly decreasing.

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Theorem. *If*

$$\int_0^{+\infty} t^{n-1} q(t)\, dt = \infty \tag{3}$$

or if (3) *fails and the solution of the second order equation*

$$v''(t) + \frac{v(t)}{(n-3)!} \int_t^{+\infty} (s-t)^{n-3} q(s)\, ds = 0 \tag{4}$$

is oscillatory, then every eventually positive solution of (1) *is either strongly increasing or strongly decreasing.*

Proof. Assume that u is an eventually positive solution of (1) and u is neither strongly increasing or strongly decreasing. Find $a \geq 0$ such that $u(t) > 0$ if $t \geq 0$. Now $u^{(n)} > 0$ on $[a, \infty)$, so $u^{(n-1)}$ is eventually one-signed. Since $u^{(n-1)}$ is eventually one-signed, $u^{(n-2)}$ is eventually one-signed. Continuing this, we see that there is $c \geq a$ such that none of $u, u', u'', \dots, u^{(n-1)}$ has any zeros in $[c, \infty)$. Let j be the largest integer such that $u^{(i)} > 0$ on $[c, \infty)$ if $i \leq j$. Since u is not strongly increasing then $j \neq n$. Now, $u^{(j)} > 0$ and $u^{(j+1)} < 0$ on $[c, \infty)$, so $u^{(j)}$ is bounded. Thus, if $j \leq k \leq n-1$, $u^{(k)} u^{(k+1)} < 0$ on $[c, \infty)$. Now, since $u^{(j-1)}$ is increasing on $[c, \infty)$

$$u^{(j)}(t) \geq \frac{1}{(n-j-1)!} \int_t^{+\infty} (s-t)^{n-j-1} q(s) u(s)\, ds$$

if $t \geq c$, and

$$\begin{aligned} u(s) &\geq \frac{1}{(j-2)!} \int_c^s (s-\xi)^{j-2} u^{(j-1)}(\xi)\, d\xi \\ &\geq \frac{1}{(j-2)!} \int_t^s (s-\xi)^{j-2} u^{(j-1)}(\xi)\, d\xi \\ &\geq \frac{u^{(j-1)}(t)}{(j-2)!} \int_t^s (s-\xi)^{j-2}\, d\xi = \frac{u^{(j-1)}(t)}{(j-1)!} (s-t)^{j-1} \end{aligned} \tag{5}$$

if $c \leq t \leq s$. Then

$$\begin{aligned} u^{(j)}(c) &\geq \frac{1}{(n-j-1)!} \int_c^{+\infty} (s-c)^{n-j-1} q(s) u(s)\, ds \\ &\geq \frac{u^{(j-1)}(c)}{(n-j-1)!(j-1)!} \int_c^{+\infty} (s-c)^{n-2} q(s)\, ds \end{aligned}$$

so (3) fails. It remains to show that (4) is nonoscillatory. Now

$$-u^{(j+1)}(t) = \frac{1}{(n-j-2)!}\int_t^{+\infty}(s-t)^{n-j-2}q(s)u(s)\,ds$$

if $t \geq c$, and this and (5) say that

$$-u^{(j+1)}(t) \geq \frac{u^{(j-1)}(t)}{(n-j-2)!(j-1)!}\int_t^{+\infty}(s-t)^{n-3}q(s)\,ds$$

if $t \geq c$. Since $(n-j-2)!(j-1)! \leq (n-3)!$, we have

$$\frac{u^{(j+1)}(t)}{u^{(j-1)}(t)} \leq -\frac{1}{(n-3)!}\int_t^{+\infty}(s-t)^{n-3}q(s)\,ds.$$

Let w be given on $[c,\infty)$ by $w = \frac{u^{(j)}}{u^{(j-1)}}$, and note that $w(t) > 0$ of $t \geq c$. Now

$$w' = \frac{u^{(j+1)}}{u^{(j-1)}} - w^2$$

so

$$w'(t) + w^2(t) \leq -\frac{1}{(n-3)!}\int_t^{+\infty}(s-t)^{n-3}q(s)\,ds \tag{6}$$

if $t \geq c$. But a clasical result of A.Wintner ([2]) and of (3) says that the existence of a positive solution of (6) implies that the solution of the equation (4) is nonoscillatory. This completes the proof.

References

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