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ON A GENERAL ITERATIVE METHOD FOR SOLVING HEREDITARY DIFFERENTIAL EQUATIONS (II)

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Abstract. The present paper is a continuation of the paper [9] in which a general iterative method for solving hereditary differential equation is considered. Using a concept of a bounded integral contractor, sufficient conditions for the existence and uniqueness of a solution of this equation are given. The fact that the Lipschitz condition is equivalent with the existence of the such contractor is proved. It is shown that the iterative procedure, utilised to prove the existence of the solution, is a special algorithm investigated in the paper [9].

1. Introduction

At the beginning, let us give in short some notions and conclusions used in the paper [9], needed in our forthcoming investigation.

Let R^k be the real k-dimensional Euclidian space and L_p^ρ, $1 \leq p \leq \infty$, be the usual space of classes of measurable functions, i.e.,

$$L_p^\rho = \left\{ \varphi \mid \ \varphi : R^+ \to R^k; \ \int_0^\infty |\varphi(t)|^p \rho(t)\, dt < \infty \right\},$$

where the function $\rho : R^+ \to R^+$, called an influence function with relaxation properties, is summable on R^+ and for every $\sigma \geq 0$ one has

$$\overline{K}(\sigma) = \operatorname*{esssup}_{s \in R} \frac{\rho(s+\sigma)}{\rho(s)} \leq \overline{\overline{K}} < \infty,$$
$$\underline{K}(\sigma) = \operatorname*{esssup}_{s \in R} \frac{\rho(s)}{\rho(s+\sigma)} < \infty.$$

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Let $X = R^k \times L_p^\rho$ be *a past-history space*, i.e., a space of elements x, $x = (\varphi(0), \varphi)$, where $\varphi \in L_p^\rho$, with the norm

$$||x||_X = \Big(|\varphi(0)|^p + \int_0^\infty |\varphi(t)|^p \rho(t)\, dt \Big)^{\frac{1}{p}} = (\,|\varphi(0)|^p + ||\varphi||_r^p\,)^{\frac{1}{p}}.$$

Obviously, X is a Banach space.

The measurable function $x : (-\infty, T] \to R^k$, T =const$\in R$, is *X-admissible* if for each $t \in (-\infty, T]$ the function x^t, called *its history up to t* and defined by $x^t(s) = x(t-s)$, $s \in R^+$, is itself element in X.

Therefore, if x is X-admissible, then $x^t = (x(t), x_r^t) \in X$ for each $t \in (-\infty, T]$.

From the definition of the norm on the space X, for each $t \geq t_0$, $t_0 \in (-\infty, T]$,

$$(1) \qquad ||x^t||_X \leq \tilde{K} \Big(|x(t)| + \overline{\overline{K}}^{\frac{1}{p}} ||x^{t_0}||_r + \Big(\int_{t_0}^t |x(u)|^p \rho(t-u)\, du \Big)^{\frac{1}{p}} \Big),$$

where $\tilde{K} = 3^{\frac{p-1}{p}} \vee 1$ (see [11]).

In the paper [9] *the hereditary differential equation*

$$(2) \qquad \dot{x}(t) = f(t, x^t)\,, \quad x^0 = \varphi\,, \quad \varphi \in X\,, \quad t \in [0, T]\,,$$

is considered, where $f : R \times X \to R^k$ is the given functional. Its solution is an X-admissible function $x \in C^1((-\infty, T]; R^k)$ with $x(0) = \varphi(0)$ and for which the equation (2) is valid on $[0, T]$. The continuity of the function $x(t)$ on $[0, T]$ implies that the function x^t is also continuous on $[0, T]$.

If the functional f is continuous in the pair of arguments and if it satisfies the a local Lipschitz condition on the second argument on a compact $\Omega \subset [0, T] \times X$, i.e., there exists a constant $L > 0$, such that for all $(t, x), (t, y) \in \Omega$,

$$(3) \qquad |f(t, x) - f(t, y)| \leq L\, ||x - y||_t,$$

where $||x||_t = \sup_{s \in [0,t]} ||x^s||_X$, then the existence of the unique solution of the equation (2), defined on an interval $(-\infty, T_1]$, $0 \leq T_1 \leq T$, is proved.

In what follows denote by $\tilde{X} = C([0, T]; X)$ a Banach space with the norm $||x|| = ||x||_T$.

2. Results

Now we consider the equation (2) using the concept of integral contractors, introduced by Altman ([1]) as a useful tool for studying different classes of deterministic equations in Banach spaces. This approach is immediately utilised to obtain very general conditions for the existence and uniqueness problems of their solutions, which includes the Lipschitz condition as a special case. There is a number of papers and books in which various types of integral contractors are considered, appropriately used to analyze some classes of functional differential and integral equations, and later stochastic differential equations (for example, [2], [7], [8], [12], [13] and in many others).

By following the ideas of Altman and Kuo ([1], [10]), we give notions and definitions of contractor theory, adapted to the equation (2).

For a fixed constant $\beta > 0$ let $\Gamma : [0,T] \times X \to R^k \times R^k$ be a continuous mapping, bounded in the sense

$$(\,\forall (t,x) \in [0,T] \times X\,)\,(\,\forall y \in R^k\,) \quad |\Gamma(t,x)\,y| \le \beta\,|y|.$$

For a fixed $x \in \tilde{X}$ let $Ax : \tilde{X} \to \tilde{X}$ be an operator, defined in the following way:

$$(\,\forall y \in \tilde{X}\,) \quad ((Ax)y)^t = (((Ax)y)(t), y_r^t),$$

where

$$((Ax)y)(t) = y(t) + \int_0^t \Gamma(s, x^s) y(s)\,ds$$

and y_r^t is an element of the space L_p^ρ.

Definition. Let the mapping Γ be defined as the above and let there exists a positive constant K, such that for any $x, y \in \tilde{X}$ and $t \in [0,T]$

$$|f(t, x^t + ((Ax)y)^t) - f(t,x^t) - \Gamma(t,x^t)\,y(t)| \le K\,||y||_t. \tag{4}$$

Then the functional f has a *bounded integral contractor* $\{I + \int_0^t \Gamma\,ds\}$.

Obviously, if f satisfies the Lipschitz condition (3), then it has the trivial integral contractor with $\Gamma \equiv 0$.

Having in mind partially the ideas from the paper [12], let us define the next norm on the space $\tilde{X}$:

$$(\,\forall x \in \tilde{X}\,) \quad |||x||| = \sup_{t\in[0,T]} \{\,||x||_t\, e^{-qt}\,\},$$

where $q > 0$ is a fixed number. Since

$$||x||_T\, e^{-qT} \le |||x||| \le ||x||_T, \tag{5}$$

the above two norms are equivalent. Therefore, $(\,\tilde{X}, |||\cdot|||\,)$ is also a Banach space.We need this fact to prove the following lemma.

Lemma 1. *Let $\Gamma : [0,T] \times X \to R^k \times R^k$ be a continuous bounded mapping. Then for every fixed $x, z \in \tilde{X}$ the operator equation*

$$((Ax)y)^t = z^t$$

has a unique solution $y \in \tilde{X}$.

Proof. For arbitrary fixed $x, z \in \tilde{X}$, let us define the operator S :

$$(\,\forall y \in \tilde{X}\,) \quad (Sy)(t) = z(t) - \int_0^t \Gamma(s, x^s)\, y(s)\, ds, \quad (Sy)_r^t = \psi_r^t,$$

where ψ_r^t is an element in L_p^ρ with $\psi_r^0 = z^0$. Clearly, $S : \tilde{X} \to \tilde{X}$. Also, from the properties of the preceding norms, for arbitrary $y_1, y_2 \in \tilde{X}$ and for $0 \le s \le t \le T$ we find

$$|(Sy_1)(s) - (Sy_2)(s)| \le \int_0^s |\Gamma(u, x^u)|\, |y_1(u) - y_2(u)|\, du$$

$$\le \beta \int_0^s ||y_1^u - y_2^u||_X\, du \le \beta \int_0^s ||y_1 - y_2||_u\, du \le \frac{\beta}{q} |||y_1 - y_2|||e^{qt}.$$

Since $(Sy_1)_r^t = (Sy_2)_r^t$ for each $t \in [0,T]$, we obtain

$$|(Sy_1)(t) - (Sy_2)(t)| = ||(Sy_1)^t - (Sy_2)^t||_X.$$

Thus

$$\sup_{s \in [0,T]} ||(Sy_1)^s - (Sy_2)^s||_X\, e^{-qt} \le \frac{\beta}{q}\, |||y_1 - y_2|||,$$

i.e., for each $t \in [0,T]$ we have

$$||Sy_1 - Sy_2||_t \le \frac{\beta}{q}\, |||y_1 - y_2|||.$$

For $q \ge 2\beta + 1$, from (5) we find

$$|||Sy_1 - Sy_2||| \le \frac{1}{2} |||y_1 - y_2|||.$$

So, by applying the Banach fixed point theorem we get that the equation $((Ax)y)^t = z^t$ has a unique solution $y \in \tilde{X}$ with the property $y_r^t = \psi_r^t$.

In the papers [1], [2], [7], [8], [10], [13] and in many others, the notion of a regularity of bounded integral contractors is introduced to prove the uniqueness of solutions of appropriate equations. Here, omiting the notion of a regularity, this problem for the equation (2) is considered.

Theorem 1. *If the continuous functional $f : R \times X \to R^k$ has a bounded integral contractor $\{I + \int_0^t \Gamma\, ds\}$, then there exists a unique solution $x \in \tilde{X}$ of the equation (2).*

Proof. Analogously to the Altman's approach and by using a paralel method with Kuo ([10]), the proof of the existence of the solution is based on two sequences of iterations: for each $t \in [0, T]$

$$x_0^t = \varphi = (\varphi(0), \varphi_r^0), \quad x_{n+1}^t = (x_{n+1}(t),\, (x_{n+1})_r^t), \quad n = 0, 1, \dots, \tag{6}$$

where

$$x_{n+1}(t) = x_n(t) - ((Ax_n)y_n)(t) = x_n(t) - y_n(t) - \int_0^t \Gamma(s, x_n^s) y_n(s)\, ds, \tag{7}$$

$$(x_{n+1})_r^t(s) = \begin{cases} x_{n+1}(t-s), & 0 \le s \le t, \\ \varphi(s-t), & s > t, \end{cases};$$

and

$$y_n^t = (y_n(t),\, (y_n)_r^t), \quad n = 0, 1, \dots, \tag{8}$$

where

$$y_n(t) = x_n(t) - \varphi(0) - \int_0^t f(s, x_n^s)\, ds, \quad (y_n)_r^t \equiv 0.$$

In accordance with the definitions of these sequences, we see that $x_n \in \tilde{X}$ and $y_n \in X$, $n = 0, 1, \dots$. Also, $||y_n^t||_X = |y_n(t)|$ for each $t \in [0, T]$. From the identity

$$y_{n+1}(t) = \int_0^t [\, f(s, x_n^s) - \Gamma(s, x_n^s) y_n(s) - f(s, x_{n+1}^s)\,]\, ds, \quad t \in [0, T],$$

and from the definition of the bounded integral contractor, after replacing x^t and y^t by x_n^t and $-y_n^t$ respectively, by repeating integrations, we find

$$|y_n(t)| \le \alpha\, K^n \,\frac{t^{n+1}}{(n+1)!}\,, \quad t \in [0, T], \tag{9}$$

where $\alpha = \max_{t \in [0,T]} |f(t, x_0^t)|$.

Therefore, $y_n(t) \to 0$ as $n \to \infty$ uniformly on $[0, T]$ and also $y_n^t \to 0$ as $n \to \infty$, uniformly on $[0, T]$ in the sense of the norm $|| \cdot ||_X$.

From (6), (7), (8) and (1), for each $t \in [0, T]$ we come to the following estimation

$$\begin{aligned} ||x_{n+1}^t - x_n^t||_X &\le B|x_{n+1}(t) - x_n(t)| \\ &\le B\alpha K^n \left(\frac{T^{n+1}}{(n+1)!} + \beta \frac{T^{n+2}}{(n+2)!} \right), \end{aligned}$$

where $B = \tilde{K}(1 + ||\rho||_L^{-})$. Since $||x_{n+1} - x_n||_T$ has the same upper bound, then $\{x_n, n \in N\}$ is the Cauchy sequence in the Banach space. So, it can be easy to conclude that its limit $x_\infty = (x_\infty(t), x_\infty^t) \in \tilde{X}$ determines the solution $x_\infty(t)$, $t \in (-\infty, T]$, of the equation (2).

To prove the uniqueness of the solution, let $x_1(t)$ and $x_2(t)$, $t \in (-\infty, T]$, be two solutions with the same initial condition, as X-admissible functions generating elements x_1 and x_2 in the space $\tilde{X}$. By taking $x = x_1$, $z = x_2 - x_1$ we obtain the operator equation

$$((Ax_1)y)^t = x_2^t - x_1^t.$$

By applying Lemma 1 we conclude that there exists its unique solution $y^t = (\, y(t), y_r^t) \in \tilde{X}$, such that for all $t \in [0, T]$,

$$y(t) + \int_o^t \Gamma(s, x_1^s) y(s)\, ds = x_2(t) - x_1(t), \tag{10}$$

and

$$((Ax_1)y)_r^t(s) = (y_r^t)(s) = \begin{cases} x_2(t-s) - x_1(t-s), & 0 \le s \le t, \\ 0, & s > t, \end{cases}$$

Since $x_1(t)$ and $x_2(t)$ are the solutions , from (10) we have

$$y(t) = \int_0^t [\, f(s, x_2^s) - f(s, x_1^s) - \Gamma(s, x_1^s) y(s)\,]\, ds.$$

Next, using (4) we find

$$\sup_{s \in [0,T]} |y(s)| \le K \int_0^t \sup_{u \in [0,s]} |y(u)|\, ds, \quad t \in [0, T].$$

By applying the Gronwall-Bellman lemma we obtain $|y(t)| = 0$ for all $t \in [0, T]$. Finally, from (10) we conclude that $x_1(t) = x_2(t)$ for all $t \in [0, T]$, what completes the proof.

Notice that for $\Gamma \equiv 0$ the sequence (7) reduces to the Picard iterations.

An error of n-th iteration can be obtained immediately, because

$$\sup_{t\in[0,T]} |x(t) - x_n(t)| \leq \sum_{k=n}^{\infty} \sup_{t\in[0,T]} |x_{k+1}(t) - x_k(t)|.$$

From (7) we get

$$|x_{k+1}(t) - x_k(t)| \leq |y_k(t)| + \beta \int_0^t |y_k(s)|\, ds, \quad t \in [0,T].$$

By using (9) we find

$$\sup_{t\in[0,T]} |x_{k+1}(t) - x_k(t)| \leq \alpha K^k \left(\frac{T^{k+1}}{(k+1)!} + \beta \frac{T^{k+2}}{(k+2)!} \right).$$

So, by summing up it is easy to show that

$$\sup_{t\in[0,T]} |x(t) - x_n(t)| \leq \frac{\alpha}{K} \cdot \frac{(KT)^n}{n!} \left(1 + \frac{\beta T}{n+1} \right) (e^{KT} - 1).$$

As we saw at the beginning, the purpose of the present paper is to show that the iterative procedure used in the proof of Theorem 1 presents a special case of a general iterative method investigated in the paper [9].

Let us have in mind the main results of the paper [9]. Together with the equation (2), we consider the sequence of hereditary differential equations

$$\dot{x}_{n+1} = F_n(t, x_{n+1}^t), \quad x_{n+1}^0 = \varphi, \quad t \in [0,T], \quad n \in N, \tag{11}$$

where $F_n : R \times X \to R^k$, $n \in N$, are given functionals. We suppose that the functionals $f(t,x)$ and $F_n(t,x)$, $n \in N$, are continuous in the pair of arguments and satisfy the local Lipschitz condition (3) on a compact $\Omega \subset [0,T] \times X$. Also, the solutions of all these equations are defined on an interval $(-\infty, T_1]$, $0 \leq T_1 \leq T$. Under these assumptions and if the following condition is satisfied,

$$\sum_{n=1}^{\infty} \sup_{t\in[0,T]} |\, F_n(t, x_n^t) - f(t, x_n^t) \,| < \infty, \tag{12}$$

it is proved that the sequence of solutions $\{x_n(t), n \in N\}$ of the equations (11) converges uniformly on $[0,T]$ to the solution $x(t)$ of the equation (2) as $n \to \infty$. Therefore, the solution $x(t)$ is approximated with the solutions $x_n(t)$, $n \in N$. Notice that in the paper [9] an error of n-th approximation is given.

This procedure for solving the equation (2) can be treated as a general iterative method called *the Z-algorithm* (analogously to the paper [14]). Since an (n+1)-th approximation $x_{n+1}(t)$ depends on the choice of the functional $F_n(t,x)$, the sequence of functionals $\{F_n(t,x), n \in N\}$ is called the *determined sequence for the Z-algotithm.* In the paper [9] constructions of concrete Z-algorithms are described, as simple forms of linearization of the functional f.

In order to prove that the sequence of iterations (7) describes a special Z-algorithm, we shall investigate a relation between the bounded integral contractor $\{I + \int_0^t \Gamma\, ds\}$ and the Lipschitz condition (3). Such relation is considered in the paper [12] for ordinary differential equation, for functional differential equation of a simpler type and for stochastic differential equation of Ito type. Notice that the proofs of the following lemma and theorem are different then ones in the paper [12].

Lemma 2. *Under the assumptions of Lemma 1, there exists a constant $\tilde{L} > 0$, independent on x and z, such that*

$$(\forall t \in [0,T]) \quad ||y||_t \le \tilde{L}\, ||z||_t.$$

Proof. From Lemma 1 it follows that for every fixed $x, z \in \tilde{X}$ the operator equation $((Ax)y)^t = z^t$ has a unique solution $y \in \tilde{X}$. So, for each $t \in [0,T]$,

$$y(t) = z(t) - \int_0^t \Gamma(s, x^s)\, ds.$$

By applying (1), for $0 \le s \le T$ we find

$$||y^s||_X \le B \sup_{u \in [0,s]} |y(u)| + \overline{\overline{K}}\, K^{-}\, ||y^0||_r, \tag{13}$$

where

$$||y^0||_r = ||z^0||_r \le ||z^0||_X \le ||z||_s.$$

Next, for $0 \le u \le s \le T$ we observe that

$$\begin{aligned} |y(u)| &\le |z(u)| + \beta \int_0^u \sup_{w \in [0,v]} |y(w)|\, dv \\ &\le \sup_{u \in [0,s]} ||z^u||_X + \beta \int_0^s \sup_{w \in [0,v]} |y(w)|\, dv, \end{aligned}$$

and therefore

$$\sup_{u\in[0,s]} |y(u)| \leq ||z||_s + \beta \int_0^s \sup_{w\in[0,v]} |y(w)|\, dv.$$

By applying the Gronwall-Bellman lemma, we get

$$\sup_{u\in[0,s]} |y(u)| \leq \gamma\, ||z||_s, \quad 0 \leq s \leq T,$$

where $\gamma = e^{\beta T} - 1$. From (13) we find

$$||y^s||_X \leq (\, B\gamma + \overline{\overline{K}}K^{-})\, ||z||_t, \quad 0 \leq s \leq t \leq T.$$

From here, putting $\tilde{L} = B\gamma + \overline{\overline{K}}K^{-}$, we obtain finally

$$||y||_t \leq \tilde{L}\, ||z||_t, \quad t \in [0, T].$$

Theorem 2. *The continuous functional* $f : R \times X \to R^k$ *has a bounded integral contractor* $\{I + \int_0^t \Gamma\, ds\}$ *if and only if it satisfies the Lipschitz condition (3) on* $[0, T] \times \tilde{X}$.

Proof. $(\Rightarrow)$ Let the functional f has a bounded integral contractor $\{I + \int_0^t \Gamma\, ds\}$. From Lemma 1, for arbitrary $x, z \in \tilde{X}$ the operator equation $((Ax)y)^t = z^t$ has a unique solution $y \in \tilde{X}$. Then, by applying Lemma 2 we obtain

$$|f(t, x^t + z^t) - f(t, x^t)|$$

$$\leq |f(t, x^t + ((Ax)y)^t) - f(t, x^t) - \Gamma(t, x^t)y(t)| + |\Gamma(t, x^t)y(t)|$$

$$\leq K\, ||y||_t + \beta\, |y(t)| \leq (\, K + \beta\,)\, ||y||_t \leq (\, K + \beta\,)\, ||z||_t.$$

$(\Leftarrow)$ Let the functional f satisfies the Lipschitz condition (3). Since $((Ax)y)^t \in \tilde{X}$ for each $x, y \in \tilde{X}$, we find

$$|f(t, x^t + ((Ax)y)^t) - f(t, x^t) - \Gamma(t, x^t)y(t)|$$

$$\leq L\, ||(Ax)y||_t + \beta\, |y(t)| \leq L\, ||(Ax)y||_t + \beta\, ||y||_t.$$

From the definition of the operator Ax, by applying (1) for $0 \le s \le t \le T$ we obtain

$$
\begin{aligned}
||((Ax)y)^s||_X &\le \tilde{K} \quad B \sup_{u \in [0,s]} |((Ax)y)(u)| + \overline{\overline{K}} \; ||y^0||_r \\
&\le \tilde{K} \quad B\,(1+\beta)\,T \sup_{u\in[0,s]} |y(u)| + \overline{\overline{K}} \sup_{u\in[0,s]} ||y^u||_X \\
&\le \tilde{K} \quad B\,(1+\beta)\,T + \overline{\overline{K}} \quad ||y||_s \le L_1 ||y||_t,
\end{aligned}
$$

where $L_1 = \tilde{K}\,[\,B\,(1+\beta)\,T + \overline{\overline{K}}\,]$. Thus

$$
\sup_{s\in[0,t]} ||((Ax)y)^s||_X = ||(Ax)y||_t \le L_1 ||y||_t.
$$

Finally, for every $x, y \in \tilde{X}$,

$$
|f(t, x^t + ((Ax)y)^t) - f(t, x^t) - \Gamma(t, x^t)y(t)| \le (\,L_1 + \beta\,)\,||y||_t,
$$

what completes the proof.

Let us remark that if the functional f satisfies the Lipschitz condition (3) on $[0,T] \times \tilde{X}$, then every continuous bounded mapping $\Gamma : [0,T] \times X \to R^k \times R^k$ defines a bounded integral contractor $\{I + \int_0^t \Gamma\, ds\}$. Its choice, obviously, determines the speed of convergence of iterations (7). It could be very interesting to compare the speeds of convergences for different integral contractors and to study how to choose the best one. However, it could be a subject of a forthcoming works.

Finally, summing up all preceding results, we are able to estabilish the main result of the present paper.

Theorem 3. *Let the continuous functional $f : R \times X \to R^k$ has a bounded integral contractor $\{I + \int_0^t \Gamma\, ds\}$. Then the sequence of iterations (7) describes the Z-algorithm of the equation (2).*

Proof. From Theorem 1 it follows that the sequence of iterations (7) converges uniformly on $[0,T]$ to the solution of the equation (2). But these iterations could be treated as the sequence of solutions of suitable hereditary differential equations. Really, from (7) and (8) we find

$$
x_{n+1}(t) = \varphi(0) + \int_0^t f(s, x_n^s)\, ds - \int_0^t \Gamma(s, x_n^s) y_n(s)\, ds,
$$
$$
t \in [0,T], \quad n = 0, 1, \ldots .
$$

Let us denote formally

$$F_n(t,x) = f(t,x_n^t) - \Gamma(t,x_n^t)y_n(t) \tag{14}$$

and let us consider the sequence of hereditary differential equations

$$\dot{x}_{n+1}(t) = F_n(t,x_{n+1}^t), \quad x_{n+1}^0 = \varphi, \quad t \in [0,T], \quad n = 0,1,\ldots. \tag{15}$$

The functionals $F_n : R \times X \to R^k$, $n = 0,1,\ldots$, are continuous and satisfy the Lipschitz condition (3) on $[0,T] \times \tilde{X}$ with the constant equal to zero. From Theorem 2 it follows that the functional f satisfies the Lipschitz condition (3) on $[0,T] \times \tilde{X}$ with a constant L; more precisely, $L = K + \beta$. So, we can consider that all functionals satisfy the Lipschitz condition (3) with the constant $K + \beta$. Naturally, all preceding properties are valid on any compact $\Omega \subset [0,T] \times \tilde{X}$, what we do not have to emphasize esspecially.

For every $n = 0,1,\ldots$ we obtain

$$\begin{aligned}\epsilon_n &= \sup_{t\in[0,T]} |F_n(t,x_n^t) - f(t,x_n^t)| \\ &= \sup_{t\in[0,T]} |\Gamma(t,x_n^t)y_n(t)| \leq \beta \sup_{t\in[0,T]} |y_n(t)|.\end{aligned}$$

From the estimation (9) we find

$$\epsilon_n = \beta\,\alpha\,K^n\,\frac{T^{n+1}}{(n+1)!}.$$

Since ${}^{\infty}_{n=0}\,\epsilon_n < \infty$, the condition (12) is also satisfied. Thus from the Theorem of the paper [9] it follows that the sequence of solutions $\{x_n(t),\ n \in N\}$ of the equations (14) converges uniformly on $(-\infty,T]$ to the solution $x(t)$ of the equation (2). Therefore, the sequence of iterations (7) is a special case of the Z-algorithm of the equation (2).

From theoretical point of view, the choice of the determined sequence (14), i.e., the choice of a bounded integral contractor Γ, makes it possible to investigate and, in the best case, to solve the equation (2) by induction.

Let us remark that the Theorem 1 could be proved indirectly by applying the Theorems 2 and 3. The existence of the solution follows immediately from the Theorem 3. It is only needed to make the estimation (9). To prove the uniqueness, it is enough to conclude that the Lipschitz condition (3) is satisfied, what follows from the Theorem 2.

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