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# A FAMILY OF SIMULTANEOUS METHODS FOR FINDING ZEROS OF ANALYTIC FUNCTIONS

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**Abstract.** In this paper a new one parameter family of simultaneous methods for finding zeros of a class of analytic function is derived. This family is obtained by applying Hansen-Patrick's third order method for solving the single equation $f(z) = 0$ to a suitable function. It is shown that all the methods of this family have fourth order of convergence.

## 1. Introduction

Let $f$ be a function of $z$ and let $\alpha$ be a fixed parameter. About twenty years ago Hansen and Patrick derived in [6] one parameter family of iteration functions for finding simple zeros of $f$ in the form

$$\hat{z} = z - \frac{(\alpha+1)f(z)}{\alpha f'(z) \pm \sqrt{f'(z)^2 - (\alpha+1)f(z)f''(z)}}. \tag{1}$$

Here $\hat{z}$ is a new approximation and $z$ is a former approximation to the desired zero. This family includes several well-known methods as Ostrowski's method ($\alpha = 0$), Euler's method ($\alpha = 1$), Laguerre's method ($\alpha = 1/(\nu - 1)$) and Halley's method ($\alpha = -1$). Also, as a limiting case ($\alpha \to \infty$), Newton's method is obtained. Except Newton's method which is quadratically convergent, all methods of the family (1) have cubic convergence to a simple zero.

In this paper we consider a class of functions $z \mapsto \Phi(z)$ analytic inside and on the simple smooth contour $\varGamma$, without zeros on $\varGamma$ and with the known number $n$ of simple zeros $\xi_1, \ldots, \xi_n$ inside $\varGamma$. This class of analytic functions will be denoted with $\Omega$. If $G$ denotes the region bounded by $\varGamma$, then $\Phi \in \Omega$ can be represented in the form

$$\Phi(z) = \exp\big(\Psi(z)\big) \prod_{j=1}^{n} (z - \xi_j) \quad (z \in G) \tag{2}$$

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(see Smirnov [12]), where $z \mapsto \Psi(z)$ is an analytic function in $G$ such that $X(z) := \exp(\Psi(z)) \neq 0$ for all $z \in G$. The analytic function $\Psi$ which appears in (2) is given by

$$\Psi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log[(w-c)^{-n}\Phi(w)]}{w-z} dw, \tag{3}$$

where $c$ is an arbitrary point in $G$ such that $\Phi(c) \neq 0$ (see [1]).

The number of zeros $n$ of $\Phi$ inside $G$ is determined by the *argument principle*

$$n = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi'(w)}{\Phi(w)} dw = \frac{1}{2\pi} \big[arg\, \Phi(w)\big]_{\Gamma}.$$

A procedure of computational interest has been proposed in [4]; the contour $\Gamma$ of the region $G$ is replaced by a polygon of vertices $A_1, \dots, A_n$ belonging to $\Gamma$.

Various methods for the simultaneous determination of zeros of analytic functions belonging to $\Omega$ have been presented in the papers [5], [7], [9] and [10]. The evaluation of $\Psi(z)$ given by (3) at some point $z = z_i$ is performed using numerical integration in the complex plane. As it was advised in [5], the contour integral (3) should be computed with satisfactory effect using trapezoidal quadrature rule. Computational aspect of the calculation of the value $\Psi(z_i)$ and the determination of the number of zeros $n$ were studied in details in the papers [5], [9], [10] so that we will not consider these subjects here.

## 2. Simultaneous methods for analytic functions

Let $P$ be a monic polynomial of degree $n$ whose zeros coincide with the zeros $\xi_1, \dots, \xi_n$ of the analytic function $\Phi \in \Omega$, that is

$$P(z) = \prod_{j=1}^{n} (z - \xi_j), \qquad \Phi(z) = \exp(\Psi(z)) P(z).$$

Let $z_1, \dots, z_n$ be $n$ pairwise distinct approximations to these zeros. Let us introduce

$$W_i = \frac{P(z_i)}{\prod_{j\neq i}(z_i - z_j)} = \frac{\exp(-\Psi(z_i))\Phi(z_i)}{\prod_{j\neq i}(z_i - z_j)} \tag{4}$$

and

$$S_{1,i} = \sum_{j\neq i} \frac{W_j}{z_i - z_j}, \quad S_{2,i} = \sum_{j\neq i} \frac{W_j}{(z_i - z_j)^2}, \quad \Upsilon_i = \sum_{j\neq i} \frac{W_j}{\xi_i - z_j}.$$

Using approximations $z_1, \ldots, z_n$, by the Lagrangean interpolation we can represent the polynomial $P$ for all $z \in \mathbb{C}$ as

$$P(z) = \prod_{j=1}^{n} (z - z_j) + \sum_{k=1}^{n} W_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - z_j). \tag{5}$$

Let us define the function $z \mapsto h_i(z)$ by

$$h_i(z) := \frac{P(z)}{\prod\limits_{j \neq i} (z - z_j)}.$$

Then, using (5),

$$h_i(z) = W_i + (z - z_i)\Big(1 + \sum_{j \neq i} \frac{W_j}{z - z_j}\Big). \tag{6}$$

Any zero $\xi_i$ of $P$ (and, consequently, of $\Phi$), is also a zero of the function $h_i(z)$. Let $I_n = \{1, \ldots, n\}$ be the index set. Starting from $h_i(z)$ we find

$$\begin{aligned} &h_i(z_i) = W_i, \quad h_i'(z_i) = 1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j}{z_i - z_j} = 1 + S_{1,i}, \\ &h_i''(z_i) = -2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j}{(z_i - z_j)^2} = -2S_{2,i} \quad (i \in I_n). \end{aligned} \tag{7}$$

To construct a new family of iterative methods for finding all zeros of $\Phi \in \Omega$ inside the region $G$, we use the idea presented by Sakurai and Petković in [11]. We apply Hansen-Patrick's formula (1) to the function $h_i(z)$ (which has the same zeros as $\Phi$ inside $G$). Substituting $f, f', f''$ that appear in (1) by $h(z_i), h'(z_i), h''(z_i)$ (given by (7)), after short arrangement we obtain a new one parameter family for the simultaneous approximation of all simple zeros of analytic function $\Phi$ inside $G$ :

$$\hat{z}_i = z_i - \frac{(\alpha + 1) W_i}{\alpha(1 + S_{1,i}) \pm \sqrt{(1 + S_{1,i})^2 + 2(\alpha + 1) W_i S_{2,i}}} \quad (i \in I_n). \tag{8}$$

**Remark.** Formula (8) contains a $\pm$ in front of the square root. Since the correction $\Delta_i = \hat{z}_i - z_i$ has to be as small in magnitude as possible, we choose the sign so that the denominator od $\Delta_i$ is *greater in magnitude*. It can be shown that for $|W_i|$ small enough (which assumes very close approximations to the zeros) the sign "+" have to be chosen. Then the main part of the iterative formula (8) is

$$\hat{z}_i = z_i - \frac{W_i}{1 + \displaystyle\sum_{j \neq i} \frac{W_j}{z_i - z_j}} \quad (i \in I_n), \tag{9}$$

which is a generalization of Börsch-Supan's method of the third order [2]. Namely, if $\Phi$ is a polynomial ($\Psi(z) \equiv 0$), then (9) reduces to Börsch-Supan's method for the simultaneous determination of all simple zeros of a polynomial.

**Example 1.** The case $\alpha = -1$ requires a limiting operation in (8). After short manipulations we get

$$\hat{z}_i = z_i - \frac{W_i(1 + G_{1,i})}{(1 + G_{1,i})^2 + W_i G_{2,i}} \quad (i \in I_n). \tag{10}$$

This formula can be derived directly by applying the classical Halley's formula to the function $h_i(z)$. Note that Ellis and Watson [3] derived the iterative formula (10) in the case of algebraic polynomials using a quite different approach.

**Example 2.** Letting $\alpha \to \infty$ in (8), we obtain the generalized Börsch-Supan's third order iterative method (9). This method can be directly obtained by applying Newton's method to the function $h_i(z)$.

## 3. Convergence analysis

Let $\hat{\varepsilon}_i = \hat{z}_i - \xi_i$ and $\varepsilon_i = z_i - \xi_i$ denote the errors in the current and previous iteration, respectively. If two complex numbers $\beta$ and $\gamma$ are of the same order in magnitude we will write $\beta = O_M(\gamma)$. In our analysis of convergence we will assume that the errors $\varepsilon_1, \dots, \varepsilon_n$ are of the same order in magnitude, that is $\varepsilon_i = O_M(\varepsilon_j)$ for any pair $i, j \in I_n = \{1, \dots, n\}$. Besides, let $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_\nu\}$ be the error with the maximal magnitude (that is $|\varepsilon| \geq |\varepsilon_i|$ $(i = 1, \dots, n)$) but still $\varepsilon = O_M(\varepsilon_i)$ for any $i \in I_n$.

**Theorem.** *Let $z_1, \dots, z_n$ be sufficiently good approximations to the zeros $\xi_1, \dots, \xi_n$ of the analytic function $\Phi \in \Omega$ in a given region $G$. Then the* *family of iterative methods (8) has the order of convergence four for any fixed and finite parameter* $\alpha$.

*Proof.* Let us introduce $u_i = W_i S_{2,i}/(1+S_{1,i})^2$. According to (4) we have

$$W_i = (z_i - \xi_i)\exp\bigl(-\Psi(z_i)\bigr)\prod_{j\neq i}\frac{z_i-\xi_j}{z_i-z_j} = \varepsilon_i \exp\bigl(-\Psi(z_i)\bigr)\prod_{j\neq i}\frac{z_i-\xi_j}{z_i-z_j},$$

so that we estimate

$$\begin{aligned}&W_i = O_M(\varepsilon_i) = O_M(\varepsilon), \quad S_{1,i} = O_M(\varepsilon), \quad S_{2,i} = O_M(\varepsilon),\\ &\Upsilon_i = O_M(\varepsilon), \quad u_i = O_M(\varepsilon^2).\end{aligned} \tag{11}$$

Assuming that $\varepsilon$ is small enough, following Remark we take the sign "+" in (8). Using the development $\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \cdots$ for $|z| \ll 1$, from (8) we obtain

$$\begin{aligned}\hat{z}_i &= z_i - \frac{(\alpha+1)W_i}{\alpha(1+S_{1,i}) + (1+S_{1,i})\sqrt{1+2(\alpha+1)u_i}}\\ &= z_i - \frac{(\alpha+1)W_i}{(1+S_{1,i})(\alpha+1+(\alpha+1)u_i + O_M(u_i^2))}.\end{aligned}$$

Hence, by the development in geometric series

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots, \quad (|z|<1), \tag{12}$$

we find

$$\begin{aligned}\hat{z}_i &= z_i - \frac{W_i}{(1+S_{1,i})\bigl(1+u_i+O_M(u_i^2)\bigr)}\\ &= z_i - \frac{W_i}{1+S_{1,i}}\Bigl(1 - \frac{W_i S_{2,i}}{(1+S_{1,i})^2} + O_M(u_i^2)\Bigr).\end{aligned} \tag{13}$$

Setting $z := \xi_i$ in (6) (with $h_i(\xi_i) = 0$) we obtain $W_i = \varepsilon_i(1+\Upsilon_i)$ so that (13) becomes

$$\hat{z}_i = z_i - \frac{W_i}{1+S_{1,i}}\left(1 - \frac{\varepsilon_i(1+\Upsilon_i)S_{2,i}}{\Bigl(1+S_{1,i}\Bigr)^2}\right) + O_M(\varepsilon^5), \tag{14}$$

where, according to (12), we estimated $W_i O_M(u_i^2) = O_M(\varepsilon^5)$. Taking $z = \xi_i$ in (6) the following fixed point relation is obtained

$$\xi_i = z_i - \frac{W_i}{1 + \sum_{j \neq i} \frac{W_j}{\xi_i - z_j}} = z_i - \frac{W_i}{1 + \Upsilon_i}. \tag{15}$$

Subtracting (15) from (14) and taking into account the estimates (11), we find

$$\hat{\varepsilon}_i = \hat{z}_i - \xi_i = \frac{W_i}{1 + \Upsilon_i} - \frac{W_i}{1 + S_{1,i}} + \frac{W_i \varepsilon_i (1 + \Upsilon_i) S_{2,i}}{(1 + S_{1,i})^3} + O_M(\varepsilon^5). \tag{16}$$

Using (12) we obtain

$$\frac{1}{1 + S_{1,i}} = 1 + O_M(\varepsilon), \ \frac{1}{(1 + S_{1,i})^3} = 1 + O_M(\varepsilon), \ \frac{1}{1 + \Upsilon_i} = 1 + O_M(\varepsilon),$$

so that

$$\begin{aligned} \frac{W_i}{1 + \Upsilon_i} - \frac{W_i}{1 + S_{1,i}} &= \frac{W_i \sum_{j \neq i} \frac{W_j (\xi_i - z_i)}{(z_i - z_j)(\xi_i - z_j)}}{(1 + \Upsilon_i)(1 + S_{1,i})} \\ &= -W_i \varepsilon_i \sum_{j \neq i} \frac{W_j}{(z_i - z_j)(\xi_i - z_j)} + \varepsilon_i^2 O_M(\varepsilon^2) + O_M(\varepsilon^5) \end{aligned}$$

and

$$\frac{W_i \varepsilon_i (1 + \Upsilon_i) S_{2,i}}{(1 + S_{1,i})^3} = W_i \varepsilon_i (1 + \Upsilon_i) S_{2,i} \big(1 + O_M(\varepsilon)\big) = W_i \varepsilon_i S_{2,i} + O_M(\varepsilon^4).$$

The two last relations give

$$\begin{aligned} &\frac{W_i}{1 + \Upsilon_i} - \frac{W_i}{1 + S_{1,i}} + \frac{W_i \varepsilon_i (1 + \Upsilon_i) S_{2,i}}{(1 + S_{1,i})^3} \\ &= -W_i \varepsilon_i \sum_{j \neq i} \frac{W_j (z_i - \xi_i)}{(z_i - z_j)^2 (\xi_i - z_j)} + O_M(\varepsilon^4) \\ &= W_i \varepsilon_i^2 O_M(\varepsilon) + O_M(\varepsilon^4) = O_M(\varepsilon^4). \end{aligned}$$

According to this from (16) there follows $\hat{\varepsilon}_i = O_M(\varepsilon^4)$, which completes the proof of theorem. $\square$

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