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# MEAN GROWTH AND $L^p$ INTEGRABILITY OF THE DERIVATIVES OF A POLYHARMONIC FUNCTION IN THE UNIT DISC

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**Abstract.** Let $f = f_0 + (1 - |x|^2)f_1 + \cdots$ ($f_j$ harmonic) be a polyharmonic function of finite degree in the unit disc $B \subset \mathbb{R}^2$. Let $X^\alpha = L^p(B, (1 - |x|)^{p\alpha-1}dx)$, $0 < p \le \infty$, $\alpha > 0$. It is proved that $\partial f/\partial x_l \in X^\alpha$ iff $|\operatorname{grad} f| \in X^\alpha$ iff $\partial f_j/\partial x_l \in X^{\alpha+j}$ for every $j$. There holds the analogous fact for higher order derivatives.

## 1. Introduction

In [4] we have considered necessary and sufficient conditions for a polyharmonic function on the unit ball $B \subset \mathbb{R}^n$ to be in the class $L^{p,q,\alpha}$ ($0 < p$, $q \le \infty$, $\alpha > 0$) consisting of those Borel functions $f$ for which

$$(1) \qquad \left\{ \int_B M_p^q(f, |x|)(1 - |x|^2)^{q\alpha-1} dx \right\}^{1/q} < \infty.$$

Here $M_p(f, \cdot)$ denote the integral means of $f$,

$$M_p(f, r) = \left\{ \int_B |f(ry)|^p d\sigma(y) \right\}^{1/p} \qquad (0 \le r < 1),$$

where $d\sigma$ is the normalized surface measure on $S = \partial B$.

In this paper we are concerned with the two-dimensional case. Thus $B$ will denote the unit disc in the two-dimensional Euclidean space. Unless specified otherwise, $\alpha$ denotes a positive real number and $p, q$ satisfy the condition $0 < p, q \le \infty$.

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A function $f \in C^\infty(B)$ is said to be polyharmonic of degree $k$ ($k$ a positive integer) if $\Delta^k f = 0$ in $B$, where $\Delta^k$ denotes the Laplace operator interated $k$ times,

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}, \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

The class of all (real-valued) functions polyharmonic of degree $k$ in $B$ is denoted by $H_k(B)$; in particular $H(B) := H_1(B)$ is the class of harmonic functions and $H_2(B)$ is the class of biharmonic functions on $B$.

A consequence of the Almansi theorem (see [1], Ch. I) is that $f$ is in $H_k(B)$ if and only if there exist functions $f_0$ to $f_{k-1}$ such that $f_j$ are harmonic and

$$f = f_0 + (1 - |x|^2) f_1 + \cdots + (1 - |x|^2)^{k-1} f_{k-1}. \tag{2}$$

Moreover $f_j$ are uniquely determined by $f$. (In [3], Hayman and Korenblum found explicit formulaes for $f_j$.)

One of the results in [4] states the following.

**Theorem A.** *Let $f$ be given by (2) where $f_j \in H(B)$. Then $f$ belongs to $L^{p,q,\alpha}$ if and only if $f_j \in L^{p,q,\alpha+j}$ for every $j$, $0 \le j \le k-1$.*

In this paper we prove the analogous result for the partial derivatives of $f$. More precisely we have

**Theorem 1.** *Let $D$ be a partial derivative of first order. Then, with the above hypotheses, $Df$ belongs to $L^{p,q,\alpha}$ if and only if $Df_j$ belongs to $L^{p,q,\alpha+j}$ for every $j$.*

This theorem does not hold in the three-dimensional case. Indeed let $f_1(x_1, x_2)$ be a harmonic function in the unit disc and let

$$f(x_1, x_2, x_3) = (1 - x_1^2 - x_2^2 - x_3^2) f_1(x_1, x_2).$$

Then $\partial f_1/\partial x_3 = 0 \in L^{p,q,\alpha}$, $f$ is biharmonic and

$$\partial f/\partial x_3 = -2x_3\, f_1(x_1, x_2).$$

It is clear that one can choose $f_1$ so that $\partial f/\partial x_3$ is not in $L^{p,q,\alpha}$.

Theorem 1 is closely related to the well known theorem of Hardy and Littlewood on harmonic conjugates (see [2] for information and references); we state it as follows.

**Theorem B.** *Let $D_i = \partial/\partial x_i$ $(i = 1, 2)$. Let $u$ be a function harmonic in $B$. Then $D_1 u$ is in $L^{p,q,\alpha}$ if and only if so is $D_2 u$.*

As a consequence of Theorem B and Theorem 1 we see that Theorem B extends to polyharmonic functions. A further consequence is the validity of Theorem 1 for higher order derivatives. See Section 3.

Theorem A is proved in [4] by means of the following lemma (Lemmas 1 and 2 in [4]). Here, for a fixed $s \geq 0$,

$$\begin{aligned} Rf &= x_1 D_1 f + x_2 D_2 f, \\ R_s f &= sf + Rf \end{aligned} \tag{3}$$

and

$$|\operatorname{grad} f| = \left((D_1 f)^2 + (D_2 f)^2\right)^{1/2}.$$

**Lemma A.** *Let $\beta > 0$. For a function $f$ polyharmonic in $B$ the following conditions are equivalent:*

(i) *$f$ is in $L^{p,q,\beta}$;*

(ii) *$R_s$ is in $L^{p,q,\beta+1}$;*

(iii) *$|\operatorname{grad} f|$ is in $L^{p,q,\beta+1}$.*

Observe that Theorem B follows immediately from Lemma A and the identity $|\operatorname{grad}(D_1 u)| = |\operatorname{grad}(D_2 u)|$, $u \in H(B)$.

In the harmonic case Lemma A is due to Hardy and Littlewood and Flett (see [2]).

## 2. Proof of Theorem 1

In order to make the proof clearer we consider the case $k = 3$. Let

$$f = f_0 + (1 - |x|^2) f_1 + (1 - |x|^2)^2 f_2,$$

where $f_0, f_1$ and $f_2$ are harmonic functions. We have

$$\begin{aligned} D_1 f = {} & D_1 f_0 - 2x_1 f_1 + (1 - |x|^2)(D_1 f_1 - 4x_1 f_2) \\ & + (1 - |x|^2)^2 D_1 f_2. \end{aligned} \tag{4}$$

The functions $x_1 f_1$ and $x_1 f_2$ are biharmonic and therefore we can write

$$x_1 f_1 = u_0 + (1 - |x|^2) u_1 \tag{5}$$

and

$$x_1 f_2 = v_0 + (1 - |x|^2) v_1, \tag{6}$$

where $v_0, v_1, u_0, u_1$ are harmonic. By differentiation we see that

$$2D_1 f_1 = \Delta(x_1 f_1) = -4u_1 - 4Ru_1,$$

and similarly for $f_2$. Hence

$$D_1 f_1 = -2R_1 u_1 \text{ and } D_1 f_2 = -2R_1 v_1. \tag{7}$$

So we can rewrite (4) as

$$\begin{aligned} D_1 f = D_1 f_0 - 2u_0 + (1-|x|^2)(-2R_2 u_1 - 4v_0) \\ + (1-|x|^2)^2(-2R_3 v_1), \end{aligned} \tag{8}$$

see (3). Now we pass to the proof. We put

$$X^\alpha = L^{p,q,\alpha}.$$

**"If" part.** Let $D_1 f_j \in X^{\alpha+j}$ $(j = 0, 1, 2)$. Then $|\operatorname{grad} f_j| \in X^{\alpha+j}$, by Theorem B, and therefore, by Lemma A,

$$f_1 \in X^\alpha \qquad \text{and} \qquad f_2 \in X^{\alpha+1}. \tag{a}$$

Also, the hypothesis together with (7) and Lemma A implies

$$-2R_3 v_1 \in X^{\alpha+2} \qquad \text{and} \qquad v_1 \in X^{\alpha+1}. \tag{b}$$

From (a), (b) and (6) we find that

$$-4v_0 \in X^{\alpha+1}. \tag{c}$$

In a similar way we show that $-2R_2 u_1 \in X^{\alpha+1}$ and $-2u_0 \in X^\alpha$. Combining this with (b), (c) and (7) we conclude that $D_1 f$ belongs to $X^\alpha$.

**"Only if" part.** Let $D_1 f \in X^\alpha$. Since the "components" of the expansion (8) are harmonic, we can apply Theorem A to obtain that

$$R_3 v_1 \in X^{\alpha+2},\ R_2 u_1 + v_0 \in x^{\alpha+1} \quad \text{and} \quad D_1 f_0 - 2u_0 \in X^\alpha. \tag{d}$$

Hence by Lemma A and (7) we get $D_1 f_2 \in X^{\alpha+2}$ and hence, by Theorem B and Lemma A, we get that $f_2 \in X^{\alpha+1}$. Since also $v_1 \in X^{\alpha+1}$, by Lemma A and (d), it follows from (6) that $v_0 \in X^{\alpha+1}$. Now the second relation in (d) shows that $R_2 u_1$ belongs to $X^{\alpha+1}$. This and (7) imply $D_1 f_1 \in X^{\alpha+1}$. Repeating the same arguments we finally see that $D_1 f_0 \in X^\alpha$, which completes the proof of Theorem 1. $\square$

## 3. Conjugate functions and higher derivatives

A pair of real-valued functions $u, v$ is called a pair of harmonic conjugates if the function $u(z) + \sqrt{-1}v(z)$, $z = x_1 + \sqrt{-1}x_2$, is holomorphic or anti-holomorphic in $z$. Clearly harmonic conjugates are harmonic functions. It is easily verified that if $f$ is harmonic, then $D_1 f$, $D_2 f$ is a pair of harmonic conjugates. The same holds for the pair $Rf$, $Tf$, where

$$Tf = x_2 D_1 f - x_1 D_2 f.$$

Therefore the following theorem generalizes the Hardy-Littlewood theorem.

**Theorem 2.** *Let $f \in H_k(B)$ for some $k$ and let $u, v$ be any two of the following four functions: $D_1 f$, $D_2 f$, $Rf$, $Tf$. Then $u \in L^{p,q,\alpha}$ if and only if $v \in L^{p,q,\alpha}$.*

*Proof.* If $u = D_1 f$ and $v = D_2 f$, then the result follows from Theorem 1 and Theorem B. Then, because of the inequalities $|Rf| \leq |\operatorname{grad} f|$ and $|Tf| \leq |\operatorname{grad} f|$, the proof reduces to proving two implications:

(i) $$Rf \in X^\alpha \Rightarrow D_1 f \in X^\alpha \qquad (X^\alpha = L^{p,q,\alpha})$$

and

(ii) $$Tf \in X^\alpha \Rightarrow D_1 f \in X^\alpha.$$

Let $Rf \in X^\alpha$. Then $D_1 Rf \in X^{\alpha+1}$, by Lemma A. Since $D_1 Rf = R_1 D_1 f$, another application of Lemma A shows that $D_1 f \in X^\alpha$. This proves (i).

Let $Tf \in X^\alpha$. The (tangential) derivative $T$ has the following property: If $\varphi$ is a radial function, then $T(\varphi f) = \varphi Tf$. From this and (2) it follows that

$$Tf = \sum_{j=0}^{k-1} (1 - |x|^2) T f_j.$$

Since $Tf_j$ and $Rf_j$ are harmonic conjugates we can apply Theorem A and the Hardy-Littlewood theorem to conclude that $Rf_j \in X^{\alpha+j}$. Hence $|x|\,|\operatorname{grad} f_j| \in X^{\alpha+j}$ because of the formula

$$(Rf)^2 + (Tf)^2 = |x|^2 |\operatorname{grad} f|^2.$$

Hence $|\operatorname{grad} f_j| \in X^{\alpha+j}$ and hence $D_1 f \in X^\alpha$, by Theorem 1. This completes the proof. $\square$

To state the next result let, for an integer $m \geq 0$,

$$\nabla_m(f) = \left( \sum_D (Df)^2 \right)^{1/2},$$

where $D$ passes through the set of all partial derivatives of order $m$. In particular $\nabla_1(f) = |\operatorname{grad} f|$ and $\nabla_0 = |f|$.

**Thorem 3.** *Let $f$ be polyharmonic in $B$ and let $Df$ be one of the $m$-th order partial derivatives of $f$. Then, if $Df \in L^{p,q,\alpha}$ then $\nabla_s(f) \in L^{p,q,\alpha}$ for all $s \leq m$.*

*Proof.* Assume that the case of the order $m-1$ has been discussed and let, for instance, $D_1^m f \in X^\alpha$ ($D_1^m = D_1 D_1 \dots$). Then $D_2 D_1^{m-1} f \in X^\alpha$, by Theorem 2, and hence $|\operatorname{grad}(D_1^{m-1} f| \in X^\alpha \subset X^{\alpha+1}$. This implies $D_1^{m-1} f \in X^\alpha$ (Lemma A). Now the result for $s < m$ follows from the induction hypothesis. (The case $m = 0$ is trivial.) If $s = m$, then successive application of Theorem 2 shows that $\nabla_s(f) \in X^\alpha$; for instance, $D_2^2 D_1^{m-2} f \in X^\alpha$ because $D_1 D_2 D_1^{m-2} f \in X^\alpha$, etc. $\square$

Finaly we return to Theorem 1 by proving that it remains true higher order derivatives.

**Theorem 4.** *Let $f \in H_k(B)$ and let $D$ be a partial derivative of any order. Then $Df \in L^{p,q,\alpha}$ if and only if $Df_j \in L^{p,q,\alpha+j}$ for all $j$.*

*Proof.* Let $Df\, X^\alpha = L^{p,q,\alpha}$, where $D$ is of order $m$. Then it is easily deduced from Theorem 3 that $T^m f \in X^\alpha$, where $T^m$ is the tangential derivative interated $m$ times. Since $T^m f = \sum (1 - |x|^2)^j T^m f_j$ we have that $T^m f_j \in X^{\alpha+j}$. Hence, by successive application of Theorem 2 together with the identity $RT = TR$, we obtain that $R^m f_j \in X^{\alpha+j}$. Now Lemma A gives that $D_1^m R^m f_j \in X^{\alpha+j+m}$. From this and the easily verified formula $D_1^m R^m = R_m^m D_1^m$ it follows that $R_m^m D_1^m f_j \in X^{\alpha+j+m}$. Hence $D_1^m f_j \in X^{\alpha+j}$ and hence $\nabla_m(f_j) \in X^{\alpha+j}$, by Theorem 3. This proves "only if" part. The proof of "if" part is similar. $\square$

**Corollary.** *If $f$ is polyharmonic in $B$, then $\nabla_m(f) \in L^{p,q,\alpha}$ if and only if $\nabla_m(f_j) \in L^{p,q,\alpha+j}$ for every $j$.*

In contrast to Theorem 1 to 4 this fact holds in any dimension.

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