FILOMAT **14** (2000), 1–8

A THEOREM ON SUBDIRECT PRODUCT OF SEMIGROUPS WITH APARTNESSES

Daniel Abraham Romano

Abstract

In this paper[1] we present a constructive version of a Bogdanović and Ćirić's lemma ([2, Lemma 1]) on subdirect product of semigroups. This version of the paper differs in some extent from the original one.

1 Introduction

1. This investigation is in the constructive mathematics ([1], [4], [5]) in the sense of Bishop, van Dalen, Heyting, Richman and Troelstra. Let $\mathbf{S} = (S, =, \neq, \cdot, 1)$ be a semigroup with apartness (A. Heyting) such that the semigroup operation is strongly extensional. Then $R_= \subseteq R^*_{\neq}$. The apartness $\neq$ is tight if $R^*_{\neq} \subseteq R_=$ (D. Scott). A relation q on S is a coequality relation if it is consistent, symmetric and cotransitive ([4]). This notion is a generalization of the notion of apartness. q is a cocongruence on S if it is compatible with the semigroup operation on S, i.e. if

$$(\forall a, b, x \in S)((ax, bx) \in q \Rightarrow (a, b) \in q \ \& \ (xa, xb) \in q \Rightarrow (a, b) \in q).$$

Let e and q be compatible congruence and cocongruence on S. Then the relation $\overline{q} = \{(x, y) \in S \times S : (x, y) \neq q\}$ is a congruence on S such that $e \subseteq \overline{q}$ and we can construct semigroups $S/(e, q)$, $S/(\overline{q}, q)$ and S/q. For these semigroups there exists a strongly extensional and embedding epimorphism $S/(e, q) \rightarrow S/q$.

2. Let S, K and Q be semigroups. If there exists a strongly extensional, embedding and injective homomorphism $\psi : S \rightarrow K \times Q$ such that $\pi_K \psi(S) = Q$, we say that S is a subdirect product of semigroups K and Q.

3. A subsemigroup K of a semigroup S is a *retract* of S if there exists a strongly extensional epimorphism $\varphi : S \rightarrow K$ such that $\varphi(x) = x$ for all

[1]Received June 6, 1998

2000 Mathematics Subject Classification: 03F55, 20M10

Key words and phrases: Semigroup with apartness, coequality relation, cocongruence, strongly extensional subset, consistent subset, subdirect product of semigroups

$x \in K$. Such epimorphism is called a *retraction*. An ideal extension S of K is a *retract extension* of K by Q if K is a retract of S and there exists a strongly extensional and embedding isomorphism $S/K \cong Q$.

4. In the second part we give some statements on semigroups with apartnesses and congruences on them. Let S be a subdirect product of a semigroup K and a semigroup Q with zero. In [2, Lemma 1] Bogdanović and Ćirić gave the following result on the retract extension:

> S is isomorphic to a semigroup which is a retract extension of K by some semigroup Q' with zero and, moreover, the semigroup Q is isomorphic to some factor-semigroup of Q'.

In this paper we shall give a constructive version of this lemma by use of cocongruences, in the sense of [4], instead of congruences. Let S be a subdirect product of a semigroup $\mathbf{K} = (K, =, \neq, \cdot, 1)$ and a semigroup $\mathbf{Q} = (Q, =, \neq, \cdot, 1, 0)$ and tight apartness such that $K \times \{0\} \subseteq S$ and let $\psi : S \to K \times Q$ be a strongly extensional, embedding and injective homomorphism such that $\pi_K\psi(S) = K$ and $\pi_Q\psi(S) = Q$. First, we shall give a proof (Lemma 4) that the set $T = \psi^{-1}(K \times Q^*)$ is a strongly extensional and consistent subset of S and a description of that subset (Corollary 4.2). We shall give also a construction of the cocongruence $\alpha(T)$ on S induced by T (Lemma 3, Corollary 4.1) and a description of that cocongruence (Corollary 4.3). Secondly, we have the following theorem: S is strongly extensional and embedding isomorphic to a semigroup which is a retract extension of K by some semigroup Q'. Moreover, Q is strongly extensional and embedding isomorphic to the semigroup $(S/\alpha)/(\beta/\alpha)$, where β is a cocongruence defined by the projection π_Q, and there exists a strongly extensional and embedding epimorphism $g : Q' \to S/\alpha$.

For notions and notations undefined here, we refer the reader to the books [1], [3] and [5] and the papers [2] and [4].

2 Preliminaries

In the next considerationsthe following results will be used:

Lemma 0 *Let q be a congruence on a semigroup S with apartness. Then the factor-sets $S/q = \{xq : x \in S\}$ and $S/(\overline{q}, q) = \{x\overline{q} : x \in S\}$ are semigroups, where*

$$x\overline{q} = y\overline{q} \Leftrightarrow (x,y)\#q, \quad x\overline{q} \neq y\overline{q} \Leftrightarrow (x,y) \in q, \quad x\overline{q} \cdot y\overline{q} = xy\overline{q},$$
$$xq = yq \Leftrightarrow (x,y)\#q, \quad xq \neq yq \Leftrightarrow (x,y) \in q, \quad xq \cdot yq = xyq$$

and the set xq is a strongly extensional subset of S for every x in S. Moreover, S/q is strongly extensional and embedding isomorphic to the semigroup $S/(\overline{q}, q)$.

Note. Let S be a semigroup with apartness and let e and q be compatible, where e is a congruence and q is a cocongruence on S. Then $e \subseteq \overline{q}$ and there are semigroups $S/(e, q)$ and $S/\overline{q}, q)$. Then there exists a strongly extensional embedding homomorphism $w : S/(e, q) \to S/(\overline{q}, q)$ defined by $w(xe) = x\overline{q}$ which is not injective, in the general case. The homomorphism w is injective if and only if $e = \overline{q}$.

Lemma 1 *Let $h : S \to R$ be a homomorphism of semigroups with apartnesses. Then the relation $q(h)$ on S defined by $(x, y) \in q(h) \Leftrightarrow h(x) \neq h(y)$ is a cocongruence on S compatible with the congruence $e(h) = \{(x, y) \in S \times S : h(x) = h(y)\}$.*

Corollary 1.1 *Let e and q be compatible congruence and cocongruence, respectively, on a semigroup S with apartness. Then there exists a strongly extensional and embedding isomorphism $(S/(e, q))/(e(w), q(w)) \to S/(\overline{q}, q)$, where $e(w) = \{(xe, ye) \in (S/(e, q))^2 : (x, y)\#q\}$ and $q(w) = \{(xe, ye) \in (S/(e, q))^2 : (x, y) \in q\}$.*

The next lemma contains an important result on cocongruences of semigroups:

Lemma 2 *Let α and β be cocongruences on a semigroup S with apartness such that $\beta \subseteq \alpha$. Then the relation β/α on S/α, defined by $\beta/\alpha = \{(x\alpha, y\alpha) \in S/\alpha \times S/\alpha : (x, y) \in \beta\}$ is a cocongruence and $(S/\alpha)/(\beta/\alpha) \cong S/\beta$ holds.*

Proof. Let $h : S/\alpha \to S/\beta$ be the mapping defined by $h(x\alpha) = x\beta$.

(i) Let $x\alpha = y\alpha$ and let $u = x\beta$. Then $(u, x) \in \beta$ and $(u, y) \in \beta$ & $(y, x) \in \beta \subseteq \alpha$. Thus we have $u \in y\beta$ because $x \in y\alpha = x\alpha$ is impossible ($x\#x\alpha$). So, $x\beta \subseteq y\beta$. Similarly, we have $y\beta \subseteq x\alpha$, therefore $x\beta = y\beta$ and h is a function.

(ii) Let $x\beta \neq y\beta$. Then $(x, y) \in \beta \subseteq \alpha$ and $x\alpha \neq y\alpha$. So, the function h is strongly extensional.

(iii) Let $x\beta$ be an element of S/β. Then $x\alpha$ is an element of S/α such that $h(x\alpha) = x\beta$ so that the function h is surjective.

(iv) We have $h(x\alpha) \neq h(y\alpha) \Leftrightarrow x\beta \neq y\beta \Leftrightarrow (x, y) \in \beta$. Thus, it follows, by Lemma 1, that the relation $\{(x\alpha, y\alpha) \in S/\alpha \times S/\alpha : (x, y) \in \beta\}$ is a cocongruence on the semigroup S/α and the mapping $\pi : S/\alpha \to (S/\alpha)/(\beta/\alpha)$ is an epimorphism. Therefore, there exists a strongly extensional, embedding homomorphism $f : (S/\alpha)/(\beta/\alpha) \to S/\beta$ such that $f \cdot \pi = h$. Let

$f((x\alpha)(\beta/\alpha)) = f((y\alpha)(\beta/\alpha))$, i.e. $h(x\alpha) = h(y\alpha)$, so $x\beta = y\beta$. Assume $u\alpha \in (x\alpha)(\beta/\alpha)$. Then $(x\alpha, u\alpha) \in \beta/\alpha$. Thus $(x\alpha, y\alpha) \in \beta/\alpha$ or $(y\alpha, u\alpha) \in \beta/\alpha$, i.e. $(x, y) \in \beta$ or $(y, u) \in \beta$. Thus it follows $u\alpha \in (y\alpha)(\beta/\alpha)$ because $y \in x\beta = y\beta$ is impossible. So, $(x\alpha)(\beta/\alpha) \subseteq (y\alpha)(\beta/\alpha)$. In a similar way we have the opposite inclusion. Therefore, the function f is injective. Finally, the homomorphism f is a strongly extensional and embedding isomorphism of semigroups.

It is well known the construction of the Rees congruence on a semigroup S induced by an ideal of S ([3]). In the next lemma we give a symmetrical construction of a cocongruence on a semigroup S with apartness induced by a strongly extensional and consistent subset of S.

Lemma 3 *Let T be a strongly extensional and consistent subset of a semigroup S with apartness. Then the relation $q(T) = \{(x, y) \in S \times S : x \neq y \,\&(x \in T \vee y \in T)\}$ is a cocongruence on S such that*

$$x \in T \Rightarrow xq(T) = \{y \in S : y \neq x\}, \quad x\#T \Rightarrow xq(T) = T.$$

Proof. (i) Let (u, v) be an arbitrary element of $q(T)$ and let z be an element of S. Then $u \neq v$ and $u \in T \vee y \in T$. Thus, the first, we have $u \neq z \vee z \neq v$, i.e. we have $(u, v) \neq (z, z)$ which means that $q(T)$ is a consistent relation. Secondly, we have, for example,

$$u \neq z \;\&\; v \in T \Rightarrow (u \neq z \;\&\; v \in T \;\&\; v \neq z) \vee (u \neq z \;\&\; v \in T \;\&\; z \in T) \\ \Rightarrow (u, z) \in q(T) \vee (z, v) \in q(T).$$

In the case $u \neq z \;\&\; u \in T$ we simply have $(u, z) \in q(T)$. Analogously, we have the implications:

$$v \neq z \,\&\, u \in T \Rightarrow (z, v) \in q(T) \vee (u, z) \in q(T) : \;\; z \neq v \,\&\, v \in T \Rightarrow (z, v) \in q(T).$$

So the relation $q(T)$ is cotransitive. It is clear that $q(T)$ is a symmetric relation.

(ii) The implication $x \in T \Rightarrow xq(T) = \{y \in S : y \neq x\}$ is clear. Let us show the second implication $x\#T \Rightarrow xq(T) = T$. Let $x\#T$ and let y be in $xq(T)$. Then $x \neq y \;\&\; y \in T$. So, $xq(T) \subset T$. Beside that, for $y \in T$ we have $x \neq y$. Therefore $(x, y) \in q(T)$ and $y \in xq(T)$. Thus $xq(T) = T$.

(iii) Let $(xu, yv) \in q(T)$, i.e. let $xu \neq yv$ and $xu \in T \vee yv \in T$. Then $x \neq y \vee u \neq v$ and $x, u \in T \vee y, v \in T$. Therefore, $(x, y) \in q(T) \vee (u, v) \in q(T)$.

Lemma 4 *Let S be a subdirect product of a semigroup $\mathbf{K} = (K, =, \neq, \cdot, 1)$ and a semigroup $\mathbf{Q} = (Q, , =, \neq, \cdot, 1, 0)$. Let $\psi : S \to K \times Q$ be a strictly* *extensional, injective and embedding homomorphism of semigroups. Then the set $\psi^{-1}(K \times Q^*)$ is a strongly extensional and consistent subset of S. If Q has not zero-divisors, then $\psi^{-1}(K \times Q^*)$ is a filter of S.*

Proof. (i) It is easily seen that the set $\psi^{-1}(K \times Q^*) = \{(a, x) \in K \times Q, x \neq 0\}$ is a consistent subset of S:

$$\begin{gathered}\psi(s' \cdot s'') \in K \times Q^* \Leftrightarrow \psi(s') \cdot \psi(s'') \in K \times Q^* \Leftrightarrow \\ (\exists a', a'' \in K)(\exists x', x'' \in Q^*)(\psi(s') = (a', x') \ \& \ \psi(s'') = \\ (a'', x'') \ \& \ \psi(s' \cdot s'') = (a' \cdot a'', x' \cdot x'') \ \& \ x' \cdot x'' \neq 0 \\ \Rightarrow (\exists a', a'' \in K)(\exists x', x'' \in Q^*)(\psi(s') = (a', x') \ \& \ \psi(s'') = (a'', x'') \\ \Rightarrow \psi(s') \in K \times Q^* \ \& \ \psi(s'') \in K \times Q^*.\end{gathered}$$

(ii) Let $t \in \psi^{-1}(K \times Q^*)$ and let s be an element of S. Then there exist $a \in K$ and $x \in Q$, $x \neq 0$, and there exist elements $b \in K$ and $y \in Q$ such that $(a, x) \in \psi(t)$ and $(b, y) \in \psi(s)$. Thus $y \neq 0 \vee y \neq x$ and $(a, x) \neq (b, y)$ or $(b, y) \in K \times Q^*$. So, the set $K \times Q^*$ is a strongly extensional subset of S.

(iii) Let Q has not zero-divisors. Then

$$\begin{gathered}\psi(s') \in K \times Q^* \ \& \ \psi(s'') \in K \times Q^* \Leftrightarrow \\ (\exists a', a'' \in K)(\exists x', x'' \in Q^*)(\psi(s') = (a', x') \ \& \ \psi(s'') = (a", x'') \Rightarrow \\ (\exists a' \cdot a'' \in K)(\exists x' \cdot x'' \in Q^*)(\psi(s' \cdot s'') = (a' \cdot a'', x' \cdot x'')) \\ \Rightarrow (\psi(s' \cdot s'') \in K \times Q^*.\end{gathered}$$

So, the consistent subset $\psi^{-1}(K \times Q^*)$ of S is a filter of S.

Corollary 4.1 *The relation α on S defined by the strongly extensional subset $T = \psi^{-1}(K \times Q^*)$ is a cocongruence on S compatible with the Rees congruence τ on S determined by the ideal $J = \psi^{-1}(K \times \{0\})$.*

Proof. Let $(x, y) \in \tau \ \& \ (y, z) \in \alpha$, i.e. let $(x = y \vee (\psi(x) \in K \times \{0\} \ \&\psi(y) \in K \times \{0\}))$ and $(y \neq z \ \& \ (\psi(y) \in K \times Q^* \vee \psi(z) \in K \times Q^*))$. We have:

(i) $x = y \ \& \ \psi(y) \neq \psi(z) \ \& \ (\psi(y) = (b, j) \in K \times Q^* \vee \psi(z) = (c, p) \in K \times Q^*) \Rightarrow \psi(x) \neq \psi(z) \ \& \ (\psi(x) = \psi(y) \in K \times Q^* \vee \psi(z) \in K \times Q^*)$.

(ii) $\psi(x) = (a, 0) \in K \times \{0\} \ \& \ \psi(y) = (b, 0) \in K \times \{0\} \ \& \ y \neq z \ \& \ \psi(z) = (c, p) \in K \times Q^* \Rightarrow$

$\psi(x) = (a, 0) \in K \times \{0\} \ \& \ \psi(z) = (c, p) \in K \times Q^* \ \&\psi(x) \neq \psi(z) \Rightarrow$
$\psi(x) \neq \psi(z) \ \& \ \psi(z) \in K \times Q^* \Rightarrow$
$(x, z) \in \alpha$.

Corollary 4.2 *$J \subseteq \{y \in S : x\#T\}$ and $T = \{y \in S : y\#J\}$.*

Proof. (i) Let x be an element of J. Then there exists an element $a \in K$ such that $\psi(x) = (a, 0)\#K \times Q$. Let y be an arbitrary element of T. Then there exist elements $b \in K$ and $j \in Q$ such that $j \neq 0$ and $(a,0) = \psi(x) \neq \psi(y) = (b,j)$. So, $x\#T$. This means $J \subseteq \{x \in S : x\#T\}$.

(ii) Let y be an arbitrary element of T. Then there exist elements $b \in K$ and $j \in Q$ such that $\psi(y) = (b,j)$ and $j \neq 0$. If x is any element of J, then there is an element $a \in K$ such that $\psi(x) = (a,0)$. Therefore, $\psi(x) = (a,0) \neq (b,j) = \psi(y)$. So, $y\#J$. This means that $T \subseteq \{y \in S : y\#J\}$. In opposite, let $y\#J$, i.e. let $(b,j) = \psi(y) \neq \psi(x) = (a,0)$ for every $a \in K$. Thus $j \neq 0$, i.e. $y \in T$.

Let S be a subdirect product of semigroups $\mathbf{K} = (K, =, \neq, \cdot, 1)$ and $\mathbf{Q} = (Q, , =, \neq, \cdot, 1, 0)$ and let $\psi : S \to K \times Q$ be a strictly extensional, injective and embedding homomorphism such that $\pi_K(\psi(S)) = K$ and $\pi_Q(\psi(S)) = Q$. Then, by Lemma 4, the set $T = \psi^{-1}(K \times Q^*)$ is a strongly extensional consistent subset of S and we can construct, by Lemma 1 and Lemma 3, congruences $\alpha(T) = \{(x,y) \in S \times S : \psi(x) \neq \psi(y)\ \&\ (x \in T \vee y \in T)\}$, $\beta(\pi_Q) = \{(x,y) \in S \times S : \pi_Q(\psi(x)) \neq \pi_Q(\psi(y))\}$ on S. We have some descriptions of these relations in the following assertion.

Corollary 4.3 *Let x be an element of S. Then*:

$$x \in T \Rightarrow x\alpha = \{y \in S : \psi(y) \neq \psi(x)\}, \quad x\#T \Rightarrow x\alpha = T;$$
$$x \in J \Rightarrow x\beta = T,\ x \in T \Rightarrow J \subseteq x\beta.$$

3 The main result

The following theorem is the main result of this paper.

Theorem 5 *Let S be a subdirect product of a semigroup K and a semigroup Q with zero and tight apartness, let $\psi : S \to K \times Q$ be a strongly extensional, injective and embedding homomorphism of semuigroups such that $K \times \{0\} \subseteq S$. Then S is strongly extensional and embedding isomorphic to a semigroup which is a retract extension of K by some semigroup Q'. Moreover, Q is a strongly extensional and embedding isomorphic to the semigroup $(S/\alpha)/(\beta/\alpha)$ and there exists a strongly extensional and embedding epimorphism $g : Q' \to S/\alpha$.*

Proof. Let $s \in S$. Then there exist elements $a \in K$ and $i \in Q$ such that $\psi(s) = (a,i)$. Then, by Lemma 4, the set $T = \psi^{-1}(K \times Q^*)$ is a strongly extensional consistent subset of S. Let $\pi_Q : K \times Q \to Q$ and $\pi_K : K \times Q \to K$ be projections, $\alpha(T) = \{(x,y) \in S \times S : \psi(x) \neq \psi(y)\ \&\ (x \in T \vee y \in T)\}$ the cocongruence on S induced by the set T, $\beta(\pi_Q) = \{(x,y) \in S \times S : \pi_Q(\psi(x)) \neq \pi_Q(\psi(y))\}$ the cocongruence on S induced by the projection π_Q

and $e(J)$ the Rees congruence on S induced by the ideal J. Beside that, we have $\beta(\pi_Q) \subset \alpha(T)$ and $\overline{\alpha(T)} \subset e(\pi_Q)$:

(i) $(x, y) \in \beta \Leftrightarrow (\exists a, b \in K)(\exists i, j \in Q)(\psi(x) = (a, i) \& \psi(y) = (b, j) \& i \neq j)$
$\Rightarrow (\exists a, b \in K)(\exists i, j \in Q)(\psi(x) = (a, i) \& \psi(y) = (b, j) \& i \neq j \& (i \neq 0 \vee 0 \neq j))$
$\Rightarrow (\exists a, b \in K)(\exists i, j \in Q)((i \in Q^*)((a, i) \neq (b, j)) \vee (j \in Q^*)((a, i) \neq (b, j))$
$\Rightarrow (\psi(x) \neq \psi(y)\ \&\ (\psi(x) \in T \vee \psi(y) \in T)$
$\Leftrightarrow (x, y) \in \alpha;$

(ii) Let $(x, y) \# \alpha(T)$. Then there exist elements $a, b \in K$ and $i, j \in Q$ such that $\psi(x) = (a, i)$ and $\psi(y) = (b, j)$. We prove that $i = j$. Suppose $i \neq j$. Then $i \neq 0 \vee 0 \neq j$ and $(x, y) \in \alpha(T)$, which is impossible. So, $\neg(i \neq j)$ and it follows that $i = j$ because the apartness in Q is tight. Therefore $(x, y) \in e(\pi_Q)$.

It is clear that $f : K \to K \times \{0\}$ is a strongly extensional and embedding isomorphism and π_K is a strongly extensional epimorphism. So, the mapping $f \cdot \pi_K \cdot \psi$ is a retraction if $K \times \{0\} \subset S$. Therefore S is strongly extensional and embedding isomorphic to a semigroup which is a retractive extension of K by a semigroup $Q' \cong S/(e(J), \alpha(T))$. Besides, there exists a strongly extensional and embedding epimorphism $S/(e(J), \alpha(T)) \to S/(\overline{\alpha(T)}, \alpha(T)) \cong S/\alpha(T)$. Moreover, since $S/\beta(\pi_Q) \cong S/(e(\pi_Q), \beta(\pi_Q)) \cong Q$, by Lemma 2 we obtain that $(S/\alpha)/(\beta/\alpha) \cong (S/(\overline{\alpha(T)}, \alpha(T))/(e(\pi_Q)/\overline{\alpha(T)}, \beta(\pi_Q)/\alpha(T)) \cong S/(e(\pi_Q), \beta(\pi_Q)) \cong S/\beta(\pi_Q) \cong Q$ whence it follows that Q is strongly extensional and embedding isomorphic to some factor-semigroup of the semigroup $S/\alpha(T)$ and there is a strongly extensional and embedding epimorphism $g : Q' \to S/\alpha(T)$.

Corollary 5.1 *There exists a strongly extensional and embedding isomorphism* $Q'/(e(g), q(g)) \cong S/\alpha(T)$.

Proof. The proof of this corollary follows immediately from Corollary 1.1.

References

[1] A. Bishop, *Foundation of Constructive Analysis*, McGraw-Hill, New York, 1967.

[2] S. Bogdanović and M. Ćirić, *Retractive nil-extensions of regular semigroups*, I, Proc. Japan Acad., Ser. A **68:5** (1992), 115–117.

[3] S. Bogdanović and M. Ćirić, *Polugrupe*, Prosveta, Niš, 1993.

[4] D.A. Romano, *Coequality relations, a survey*, Bull. Soc. Math. Banja Luka **3** (1996), 1–35.

[5] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction*, North-Holland, Amsterdam, 1988.

Faculty of Natural Sciences and Mathematics
Mladena Stojanovica 2
78000 Banja Luka
Republic Srpska, Bosnia and Herzegovina