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A PRESERVING PROPERTY OF A LIBERA TYPE OPERATOR

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions wich are regular in the unit disc U and

$$A = \{f \in \mathcal{H}(U),\ f(0) = 0,\ f^{'}(0) = 1\}.$$

Let the integral operator $L_a : A \to \mathcal{H}(U)$ as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(z)\, t^{a-1} dt \qquad a \in \mathbf{C},\ Re\, a \geq 0\,.$$

In the case $a = 1$ this operator was introduced by R.J. Libera and it was studied by many authors in different general cases. In the form (1) was used first time by N.N. Pascu.

The purpose of this note[1] is to show that the n-uniform starlike functions of order γ and type α and the n-uniform close to convex functions of order γ and type α are preserved by the operator in form (1).

2 Preliminary results

Let D^n the Sălăgean differential operator defined as:

Definition 2.1 $D^n : A \to A$, $n \in \mathbf{N}$ *and*

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf^{'}(z), \\ D^n f(z) &= D\left(D^{n-1} f(z)\right)\,. \end{aligned}$$

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Definition 2.2 *[2] Let $f \in A$, we say that f is n-uniform starlike function of order γ and type α if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| + \gamma, \quad z \in U$$

where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in \mathbf{N}$. We denote this class with $US_n(\alpha, \gamma)$.

Remark 2.1 *Geometric interpretation: $f \in US_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1}f(z)}{D^n f(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha.\gamma}$, where $\Delta_{\alpha.\gamma}$ is a eliptic region for $\alpha > 1$, a parabolic region for $\alpha = 1$, a hiperbolic region for $0 < \alpha < 1$, the half plane $u > \gamma$ for $\alpha = 0$.*

Definition 2.3 *[1] Let $f \in A$, we say that f is n-uniform close to convex function of order γ and type α in respect to the funtion n-uniform starlike of order γ and type α g(z), where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| + \gamma, \quad z \in U$$

where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in \mathbf{N}$. We denote this class with $UCC_n(\alpha, \gamma)$.

Remark 2.2 *Geometric interpretation: $f \in UCC_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1}f(z)}{D^n g(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha.\gamma}$, where $\Delta_{\alpha.\gamma}$ is a eliptic region for $\alpha > 1$, a parabolic region for $\alpha = 1$, a hiperbolic region for $0 < \alpha < 1$, the half plane $u > \gamma$ for $\alpha = 0$.*

The next two theorems are results of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [3], [4], [5]).

Theorem 2.1 *Let h be convex in U and $Re\,[\beta h(z) + \gamma] > 0, z \in U$. If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \text{ then } p(z) \prec h(z) \ .$$

Theorem 2.2 *Let q be convex in U and $j : U \to \mathbf{C}$ with $Re[j(z)] > 0$. If $p \in \mathcal{H}(U)$ and p satisfied $p(z) + j(z) \cdot zp'(z) \prec q(z)$, then $p(z) \prec q(z)$.*

3 Main results

Theorem 3.1 *If* $F(z) \in US_n(\alpha, \gamma)$*, with* $\alpha \geq 0$ *and* $\gamma > 0$*, then* $f(z) = L_a F(z) \in US_n(\alpha, \gamma)$ *with* $\alpha \geq 0$ *and* $\gamma > 0$*.*

Proof. We know that $F(z) \in US_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1}F(z)}{D^n F(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha.\gamma}$
By differentiating (1) we obtain:

$$(1+a)F(z) = af(z) + zf^{'}(z).$$

By means of the application of the linear operator D^{n+1} we obtain:

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf^{'}(z)),$$

or

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z).$$

Similarly, by means of the application of the linear operator D^n we obtain:

$$(1+a)D^n F(z) = aD^n f(z) + D^{n+1}f(z)).$$

Thus $$\frac{D^{n+1}F(z)}{D^n F(z)} = \frac{D^{n+2}f(z) + aD^{n+1}f(z)}{D^{n+1}f(z) + aD^n f(z)} =$$

(2) $$= \frac{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \cdot \frac{D^{n+1}f(z)}{D^n f(z)} + a \cdot \frac{D^{n+1}f(z)}{D^n f(z)}}{\frac{D^{n+1}f(z)}{D^n f(z)} + a}.$$

With notation $\frac{D^{n+1}f(z)}{D^n f(z)} = p(z)$, where $p(z) = 1 + p_1 z + \cdots$, we have

$$zp^{'}(z) = z \cdot \left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{'} = \frac{z(D^{n+1}f(z))^{'} \cdot D^n f(z) - D^{n+1}f(z) \cdot z(D^n f(z))^{'}}{(D^n f(z))^2} =$$

$$= \frac{D^{n+2}f(z) \cdot D^n f(z) - (D^{n+1}f(z))^2}{(D^n f(z))^2}$$

$$\frac{1}{p(z)} \cdot zp^{'}(z) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - p(z).$$

From here we obtain

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp^{'}(z).$$

Thus from (2) we obtain:

$$\begin{aligned} \frac{D^{n+1}F(z)}{D^nF(z)} &= \frac{p(z)\cdot\left(zp^{'}(z)\cdot\frac{1}{p(z)}+p(z)\right)+a\cdot p(z)}{p(z)+a} = \\ &= p(z)+\frac{1}{p(z)+a}\cdot zp^{'}(z). \end{aligned} \tag{3}$$

If we consider $h \in \mathcal{H}_u(U)$, with $h(0) = 1$, which maps the unit disc into the convex domain included in right half plane $\Delta_{\alpha.\gamma}$, then from $\frac{D^{n+1}F(z)}{D^nF(z)}$ take all values in $\Delta_{\alpha.\gamma}$, using (3) we obtain:

$$p(z) + \frac{1}{p(z)+a} \cdot zp^{'}(z) \prec h(z),$$

where, from her construction, we have $Re\, h(z) > 0$ and from hypothesis $Re\, a \geq 0$. From here folows that $Re(h(z) + a) > 0$. In this conditions from theorem (2.1) we obtain $p(z) \prec h(z)$. From here follows that $\dfrac{D^{n+1}f(z)}{D^nf(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha.\gamma}$, or
$f(z) = L_aF(z) \in US_n(\alpha, \gamma)$, with $\alpha \geq 0$ and $\gamma > 0$.

Theorem 3.2 *If $F(z) \in UCC_n(\alpha, \gamma)$, in respect to the funtion n-uniform starlike of order γ and type α $G(z)$, with $\alpha \geq 0$ and $\gamma > 0$, then*
$f(z) = L_aF(z) \in UCC_n(\alpha, \gamma)$ in respect to the funtion n-uniform starlike of order γ and type α ,see theorem (3.1), $g(z) = L_aG(z)$ with $\alpha \geq 0$ and $\gamma > 0$.

Proof. We know that $F(z) \in UCC_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1}F(z)}{D^nG(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha.\gamma}$
By differentiating (1) we obtain:
$(1 + a)F(z) = af(z) + zf^{'}(z)$, and
$(1 + a)G(z) = af(z) + zg^{'}(z)$.
By means of the application of the linear operator D^{n+1} we obtain:

$$(1 + a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf^{'}(z)),$$

or

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z).$$

Similarly, by means of the application of the linear operator D^n we obtain:

$$(1+a)D^{n}G(z) = aD^{n}g(z) + D^{n+1}g(z)).$$

With simple calculation we obtain:

$$(4) \qquad \frac{D^{n+1}F(z)}{D^nG(z)} = \frac{\dfrac{D^{n+2}f(z)}{D^{n+1}g(z)} \cdot \dfrac{D^{n+1}g(z)}{D^ng(z)} + a \cdot \dfrac{D^{n+1}f(z)}{D^ng(z)}}{\dfrac{D^{n+1}g(z)}{D^ng(z)} + a}.$$

With notation $\dfrac{D^{n+1}f(z)}{D^ng(z)} = p(z)$, and $\dfrac{D^{n+1}g(z)}{D^ng(z)} = h(z)$ by a similar calculus as the above theorem, it follows that:

$$\frac{D^{n+2}f(z)}{D^{n+1}g(z)} = p(z) + \frac{1}{h(z)} \cdot zp^{'}(z).$$

Thus from (4) we obtain:

$$(5) \qquad \frac{D^{n+1}F(z)}{D^nG(z)} = p(z) + \frac{1}{h(z)+a} \cdot zp^{'}(z).$$

If we consider q convex in unit disc U, which maps the unit disc into the convex domain included in right half plane $\Delta_{\alpha.\gamma}$, then from $\frac{D^{n+1}F(z)}{D^nG(z)}$ take all values in $\Delta_{\alpha.\gamma}$, using (5) we obtain:

$$p(z) + \frac{1}{h(z)+a} \cdot zp^{'}(z) \prec q(z),$$

where, from her construction, we have $Re\, h(z) > 0$ and from hypothesis $Re\, a \geq 0$. From here folows that $Re \dfrac{1}{h(z)+a} > 0$. In this conditions from theorem (2.2) we obtain $p(z) \prec q(z)$. From here follows that $\dfrac{D^{n+1}f(z)}{D^ng(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha.\gamma}$, or $f(z) = L_aF(z) \in UCC_n(\alpha,\gamma)$, in respect to $g(z) = L_aG(z) \in US_n(\alpha,\gamma)$ with $\alpha \geq 0$ and $\gamma > 0$.

4 Some particular casses

1. From Theorem 3.1, for $n = 1$, we obtain that the integral operator (1) preserved the class $US^c(\alpha, \gamma)$, with $\gamma > 0$, of uniform convex of type α and of order γ functions, introduced by I. Magdaş.
2. From Theorem 3.1, for $n = 1$, $\gamma = 0$ we obtain that the integral operator (1) preserved the class $US^c(\alpha)$, of uniform convex of type α functions, introduced by S. Kanas and A. Visniowska.
3. From Theorem 3.1, for $n = 1$, $\gamma = 0$, $\alpha = 1$, we obtain that the integral operator (1) preserved the class US^c, of uniform convex functions, introduced by A.W. Goodman, and studied by W. Ma and D. Minda.
4. From Theorem 3.1, for $n = 1$, $\alpha = 1$, we obtain that the integral operator (1) preserved the class $US^c[\gamma]$, with $\gamma > 0$, of uniform convex of order γ functions, introduced by F. Ronning.
5. From Theorem 3.1, for $n = 0$, $\alpha = 1$, we obtain that the integral operator (1) preserved the class $SP\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$, introduced by F. Ronning.
6. From Theorem 3.2, for $\gamma = 0$, we obtain that the integral operator (1) preserved the class $UCC_n(\alpha)$, of n-uniform close to convex of type α in respect to a n-uniform starlike of type α functions, introduced by D. Blezu.

References

[1] D. Blezu, *On the n-uniform close to convex functions with respect to a convex domain*, Demonstratio Mathematica (to appear).

[2] I. Magdaş, *Doctoral thesis*, University "Babeş-Bolyai", Cluj-Napoca, 1999.

[3] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Mich. Math. **28** (1981), 157–171.

[4] S.S. Miller and P.T. Mocanu, *Univalent solution of Briot-Buoquet differential equations* , J. Differential Equations **56** (1985), 297–308.

[5] S.S. Miller and P.T. Mocanu, *On some classes of first-order differential subordinations*, Mich. Math. **32** (1985), 185–195.

[6] Gr. Sălăgean, *On some classes of univalent functions*, Seminar of geometric function theory, Cluj-Napoca, 1983.

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