Cantor dust by AIFS

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Abstract

Affine invariant Iterated Function Systems (AIFSs), introduced earlier by the authors, are further investigated from the point of view of ability to generate sets of Cantor-type. Conditions upon which the AIFS is totally disconnected are discussed [1].

1 Introduction: IFS and AIFS

An IFS, or Iterated Function System, in the metric space $(\mathbf{X}, d)$ is the system

$$\Sigma = \{\mathbf{X}; w_1, \ldots, w_n\}, \quad n \geq 2, \tag{1}$$

where each w_i denotes a contraction in $(\mathbf{X}, d)$ with the Lipschitz factor $s_i < 1$ [1]. The *attractor* of the IFS Σ is a compact subset of $\mathbf{X}$ uniquely attached to Σ by the following procedure. The set of all nonempty compact subsets of $\mathbf{X}$, say $\mathcal{H}[\mathbf{X}]$, with the Hausdorff metric induced by d, say h_d, gives the complete metric space ($\mathcal{H}[\mathbf{X}]$, h_d). Here the *Hutchinson operator* associated to Σ, namely

$$W_\Sigma(\cdot) = \cup_{i=1}^{n} w_i(\cdot), \tag{2}$$

is a contraction, with the Lipschitz factor $s = \max_i\{s_i\} < 1$. The fixed point of W_Σ in ($\mathcal{H}[\mathbf{X}]$, h_d) is called the *attractor* of Σ. By definition, it satisfies

$$\mathrm{att}(\Sigma) = W_\Sigma\big(\mathrm{att}(\Sigma)\big) = \lim_{k\to\infty} (W_\Sigma)^k(G), \quad G \in \mathcal{H}[\mathbf{X}]. \tag{3}$$

Having, mostly, a fractional Hausdorff dimension, att(Σ) is called a *fractal set.*

In this paper we use an alternative form IFS, the AIFS or affine invariant Iterated Function System, that was introduced by ourselves in [5], [7], and also was dealt with in [6], [8], [9]. The metric space where we work is a real m-dimensional space ($\mathbb{R}^m$, d) with $m \geq 2$, but we omit to mention the distance d whenever this is not quite relevant. In ($\mathbb{R}^m$, d) an AIFS is a system defined by an $(m-1)$-dimensional simplex along with two or more real square m-dimensional *row-stochastic* matrices, according to the following definition.

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Definition 1 (AIFS) *With* $m, n \geq 2$, *let* $\Delta_{\mathbf{T}} \subset \mathbb{R}^{m-1}$ *be a non-degenerate closed simplex with vertices* $\mathbf{T} = [\mathbf{T}_1, \ldots, \mathbf{T}_m]^T$, *and let* $\{S_i\}_{i=1}^n$, *be a set of real square nonsingular row-stochastic matrices of order* m. *The system*

$$\Omega_{\mathbf{T}} = \{\mathbf{T};\, S_1, \ldots, S_n\}, \quad n \geq 2$$

is the corresponding AIFS (affine invariant IFS) in $\mathbb{R}^m$. *If all the linear mappings associated with the matrices* S_i *are contractions in* $(\mathbb{R}^m, d)$, $\Omega_{\mathbf{T}}$ *is said hyperbolic and it has the unique attractor* $\mathrm{att}(\Omega_T) \subset \mathbb{R}^{m-1}$.

2 Canonical AIFS

First we state some terminology and basic facts about AIFS, then we will establish a connection between any affine IFS in a real multi-dimensional metric space and a corresponding AIFS.

As customary, we identify a point in the m-dimensional affine space with its cartesian coordinates vector. We denote, for instance, by $\{\mathbf{e}_i = [\delta_{ij}]_{j=1}^m\}_{i=1}^m$ the orthonormal basis of $\mathbb{R}^m$ and by $\mathbf{E} = [\mathbf{e}_1 \ldots \mathbf{e}_m]^T$ the m-vector having unit points of the corresponding affine space as its components. By $\Delta_{\mathbf{T}}$ we denote the convex hull of the points of $\mathbb{R}^m$ which are components of the vector $\mathbf{T} = [\mathbf{T}_1 \ldots \mathbf{T}_m]^T$, namely the simplex $\Delta_{\mathbf{T}} = \mathrm{conv}(\mathbf{T}_1, \ldots, \mathbf{T}_m)$. Identifying a simplex by its vertices, we also refer to "the simplex $\mathbf{T}$" as "the simplex $\Delta_{\mathbf{T}}$". Furthermore, we denote by $\mathbf{V}$ the affine hull of the unit points $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ (and write $\mathbf{V} = \mathrm{aff}(\mathbf{e}_1, \ldots, \mathbf{e}_m) \subset \mathbb{R}^m$), or the hyperplane satisfying $\sum_{i=1}^m x_i = 1$. This is an $(m-1)$-dimensional subspace of $\mathbb{R}^m$, orthogonal to the direction of $\mathbf{e} = [1\,1\,\ldots\,1]^T$. And we denote by $\mathbb{R}_\perp^{m-1}$the hyperplane given by the equation $x_m = 1$, a subspace orthogonal to the the vector $\mathbf{e}_m = [0\,0 \ldots 0\,1]^T$.

Definition 2 (Canonical simplex) *Let* $\{\mathbf{e}_i\}_{i=1}^m$ *be the orthonormal basis of* $\mathbb{R}^m$ *and let* $\mathbf{E} = [\mathbf{e}_1 \ldots \mathbf{e}_m]^T$. *The simplex* $\Delta_{\mathbf{E}} \subset \mathbf{V}$ *is the canonical simplex of* $\mathbb{R}^m$.

Remark 1 *It is straightforward that for the points in* $\mathbf{V}$ *barycentric coordinates with respect to the simplex* $\Delta_{\mathbf{E}}$ *coincide with cartesian coordinates in* $\mathbb{R}^m$.

Let $\mathcal{M}_n$ denote the family of square matrices of order n, and $\mathcal{S}_n$ denote the one of row stochastic matrices, namely matrices whose rows sum up to one. Given $S \in \mathcal{M}_m$, we call the linear mapping $\mathcal{L} : \mathbb{R}^m \to \mathbb{R}^m$ such that $\mathcal{L}(\mathbf{x}) = S^T\mathbf{x}$, the *linear mapping associated with* S. We denote by Λ the set of all linear mappings $\mathcal{L} : \mathbb{R}^m \to \mathbb{R}^m$, and by Λ_S its subset consisting of the mappings associated with matrices in $\mathcal{S}_n$. Also, we denote by Φ the set of affine mappings of $\mathbb{R}^{m-1}$, namely mappings $w : \mathbf{x} \in \mathbb{R}^{m-1} \mapsto A\,\mathbf{x} + \mathbf{b} \in \mathbb{R}^{m-1}$, with $A \in \mathcal{M}_{m-1}$, $\mathbf{b} \in \mathbb{R}^{m-1}$.

Remark 2 $\mathcal{L}(\mathbf{V}) = \mathbf{V}$, $\forall\, \mathcal{L} \in \Lambda_S$. *In other words, the subspace* $\mathbf{V}$ *is invariant for any linear transformation of* $\mathbb{R}^m$ *associated with a row stochastic matrix.*

Now, it is known that for any matrix M a left eigenvector of M is orthogonal to any right eigenvector of M corresponding to a different eigenvalue, and that the right eigenvectors of M^T are left eigenvectors of M. Also it is known that any matrix $S \in \mathcal{S}_m$ admits the eigenvalue $\lambda = 1$ and that the right eigenvector of S associated with $\lambda = 1$ is $\mathbf{e} = [\,1\,1\,\ldots\,1]^T \in \mathbb{R}^m$. We deduce from this that given any $S \in \mathcal{S}_m$ having the spectrum $\{\lambda_i\}_{i=1}^m$ with $\lambda_m = 1$, provided that this eigenvalue is *simple*, all the right eigenvectors of S^T corresponding to the (other) eigenvalues $\lambda_1, \ldots, \lambda_{m-1}$ are orthogonal [1] to $\mathbf{e}$ and therefore, denoting them by $\{\mathbf{v}_i\}_{i=1}^{m-1}$, we have $\mathbf{v}_i \subset \mathbf{V}$ $(i = 1, \ldots, m-1)$. Furthermore, if they are linearly independent, they form a basis in $\mathbf{V}$, and therefore

Remark 3 *Given* $S \in \mathcal{S}_m$, *having linearly independent left eigenvectors* $\{\mathbf{v}_i^T\}_{i=1}^{m-1}$ *corresponding to the non-unit eigenvalues* $\{\lambda_i\}_{i=1}^{m-1}$, *any vector* $\mathbf{u} \subset \mathbf{V}$ *admits the representation* $\mathbf{u} = \sum_{i=1}^{m-1} \alpha_i \mathbf{v}_i$, *with* $\alpha_i \in \mathbf{R}$, $\forall\, i$.

We have tools, now, to introduce the AIFS, consistently with the general IFS theory recalled earlier.

Definition 3 (Canonical AIFS) *Given n m-dimensional row stochastic matrices* $\{\, S_i\}_{i=1}^n$ $(n \geq 2)$*, the corresponding canonical AIFS is defined by the relation*

$$\Omega = \{\mathbf{E};\, S_1, \ldots, S_n\} \equiv \{\mathbf{V};\, \mathcal{L}_1, \ldots, \mathcal{L}_n\}. \tag{4}$$

The compact set $\mathrm{att}(\Omega) \subset \mathbf{V}$ *being the attractor of the IFS*

$$\Omega = \{\mathbf{V};\, \mathcal{L}_1, \ldots, \mathcal{L}_n\}$$

is, by definition, the corresponding (canonical) attractor.

Notice that the attractor of a canonical AIFS of dimension m lies in the space spanned by the vertices of the canonical simplex of $\mathbb{R}^m$, namely a space having dimension $m-1$. In fact, the requirement that $S_i \in \mathcal{S}_m$, $\forall\, i$, guarantees that all the iterates stay in $\mathbf{V}$ (see Remark 2), so the AIFS actually "works" in $\mathbf{V}$.

We will show how to introduce a canonical AIFS corresponding to any given affine IFS. First, introduce the (immersion) map $i\,:\, \mathbf{x} \in \mathbb{R}^{m-1} \mapsto [\mathbf{x}^T |\, 1]^T = [x_1, x_2, \ldots, x_{m-1}, 1]^T \in \mathbb{R}_\perp^{m-1}$, which is obviously one to one and, so to say, embeds the space $\mathbb{R}^{m-1}$ into $\mathbb{R}^m$ by "laying it over" the hyperplane $x_m = 1$. Then, notice that, even though the (orthogonal) projection from $\mathbb{R}^m$ to $\mathbb{R}_\perp^{m-1}$, namely the map $\mathrm{proj}_\perp\,:\, [x_1, \ldots, x_{m-1}, x_m]^T \in \mathbb{R}^m \mapsto [x_1, \ldots, x_{m-1}, 1]^T \in \mathbb{R}_\perp^{m-1}$ is not invertible in $\mathbb{R}^m$, its restriction to $\mathbf{V}$ is such (it is a one to one map from $\mathbf{V}$ to $\mathbb{R}_\perp^{m-1}$). Then, introducing the block triangular matrices of $\mathcal{M}_m$

$$S_p = \begin{bmatrix} I_{m-1} & \vdots & \mathbf{1} \\ \cdots & \cdots & \cdots \\ \mathbf{0}^T & \vdots & 1 \end{bmatrix}, \qquad S_l = S_p^{-1} = \begin{bmatrix} I_{m-1} & \vdots & -\mathbf{1} \\ \cdots & \cdots & \cdots \\ \mathbf{0}^T & \vdots & 1 \end{bmatrix}, \tag{5}$$

[1]Indeed, this can also be checked directly noticing that $\forall\, i \in \{1, \ldots, m-1\}$, $\lambda_i \langle \mathbf{v}_i, \mathbf{e}\rangle = \langle \lambda_i \mathbf{v}_i, \mathbf{e}\rangle = (\lambda_i \mathbf{v}_i)^* \mathbf{e} = (S^T \mathbf{v}_i)^* \mathbf{e} = (\mathbf{v}_i^* S)\mathbf{e} = \mathbf{v}_i^*(S\mathbf{e}) = \mathbf{v}_i^* \mathbf{e} = \langle \mathbf{v}_i, \mathbf{e}\rangle$, where $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^* \mathbf{v}$ denotes the usual scalar product of complex-valued vectors $\mathbf{u}$ and $\mathbf{v}$. Now, if S is a regular matrix so that $\lambda_i \neq 0$ $(i = 1, \ldots, m)$, $\lambda_i \langle \mathbf{v}_i, \mathbf{e}\rangle = \langle \mathbf{v}_i, \mathbf{e}\rangle$ yields that if $\lambda_i \neq 1 = \lambda_m$, it must be $\langle \mathbf{v}_i, \mathbf{e}\rangle = 0$.

where $\mathbf{0} = [0, \ldots, 0]^T \in \mathbb{R}^{m-1}$, $\mathbf{1} = [1, \ldots, 1]^T \in \mathbb{R}^{m-1}$ and $I_{m-1} \in \mathcal{M}_{m-1}$ is the unit matrix, we see that the linear maps associated to S_p and S_l, namely $\mathcal{L}_p \in \Lambda$ and $\mathcal{L}_l = \mathcal{L}_p^{-1} \in \Lambda$, respectively map $\mathbf{V} \subset \mathbb{R}^m$ into $\mathbb{R}_\perp^{m-1} \subset \mathbb{R}^m$ and vice-versa. More precisely, if $\mathbf{x} = [x_1, x_2, \ldots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i]^T \in \mathbf{V}$ and $\mathbf{x}_p = [x_1, x_2, \ldots, x_{m-1}, 1]^T \in \mathbb{R}_\perp^{m-1}$, then we see that $\mathcal{L}_p(\mathbf{x}) = S_p^T \mathbf{x} = \mathbf{x}_p$ and $\mathcal{L}_l(\mathbf{x}_p) = S_l^T \mathbf{x}_p = \mathbf{x}$. Therefore the restriction of $\mathcal{L}_p$ to $\mathbf{V}$ is the same as the restriction of $\mathrm{proj}_\perp$ to $\mathbf{V}$, but the first one is linear and is invertible in $\mathbb{R}^m$ (its inverse being $\mathcal{L}_l$) while the latter, as we said, is not. We call $\mathcal{L}_p$ a *linear projection* (from $\mathbf{V}$ to $\mathbb{R}_\perp^{m-1}$) and its inverse, $\mathcal{L}_l$, a (linear) *lifting*. Finally, we can compose $\mathcal{L}_p$ and $\mathcal{L}_l$ by the immersion map i introduced earlier, so to obtain the invertible (projection) map $\pi\,(= i^{-1}\mathcal{L}_p) : \mathbf{V} \to \mathbb{R}^{m-1}$ with its inverse $\pi^{-1}\,(= \mathcal{L}_l\, i) : \mathbb{R}^{m-1} \to \mathbf{V}$. As it is easy to check directly, π preserves convex combinations, so that the following definitions make sense.

Definition 4 (Standard simplex) *The standard simplex of $\mathbb{R}^{m-1}$, denoted by $\Delta_0 \subset \mathbb{R}^{m-1}$, is the projection, by means of π, of the canonical simplex of $\mathbb{R}^m$.*

Remark 4 *Since π preserves convex combinations, we have $\Delta_0 = \Delta_{\mathbf{E}_0}$ where $\mathbf{E}_0 = \pi(\mathbf{E}) = [\,\pi(\mathbf{e}_1)\ \pi(\mathbf{e}_2)\ \ldots\ \pi(\mathbf{e}_m)]^T = [\,\mathbf{e}'_1\ \mathbf{e}'_2\ \ldots \mathbf{e}'_{m-1}\ \mathbf{0}\,]^T$, with the notation $\{\mathbf{e}_i = [\delta_{ij}]_{j=1}^m\}_{i=1}^m$ and $\{\mathbf{e}'_i = [\delta_{ij}]_{j=1}^{m-1}\}_{i=1}^{m-1}$. In fact, by definition*

$$\Delta_0 = \pi(\Delta_{\mathbf{E}}) = \Delta_{\pi(\mathbf{E})} = \Delta_{\mathbf{E}_0}\ , \quad \Delta_{\mathbf{E}} = \pi^{-1}(\Delta_0) = \mathcal{L}_l i\,(\Delta_0). \tag{6}$$

Definition 5 (Standard AIFS) *Given n row stochastic matrices of order m, $\{S_i\}_{i=1}^n$ $(n \geq 2)$, denoting by $\mathbf{E}_0$ the standard simplex of $\mathbb{R}^{m-1}$, the AIFS*

$$\Omega_0 = \{\mathbf{E}_0;\, S_1, \ldots, S_n\} \equiv \{\mathbb{R}^{m-1};\, \pi\,\mathcal{L}_1\,\pi^{-1}, \ldots, \pi\,\mathcal{L}_n\,\pi^{-1}\} \tag{7}$$

is the corresponding standard AIFS and the attractor $\mathrm{att}(\Omega_0) \subset \mathbb{R}^{m-1}$ *is called the corresponding standard attractor.*

Definition 6 *Let $w \in \Phi$ and $\mathcal{L} \in \Lambda_S$. The mapping w is said the (orthogonal) projection of $\mathcal{L}$ on $\mathbb{R}^{m-1}$, and is denoted by $w = \Pi(\mathcal{L})$, iff*

$$\pi\big(\mathcal{L}(\mathbf{x})\big) = w\big(\pi(\mathbf{x})\big), \quad \forall\, \mathbf{x} \in \mathbf{V}. \tag{8}$$

Theorem 1 *For any $w \in \Phi$ there is one and only one $\mathcal{L} \in \Lambda_S$ such that $w = \Pi(\mathcal{L})$. More precisely, if $w : \mathbf{x} \in \mathbb{R}^{m-1} \mapsto A\mathbf{x} + \mathbf{b} \in \mathbb{R}^{m-1}$ and $\mathcal{L} : \mathbf{x} \in \mathbb{R}^m \mapsto S^T\mathbf{x} \in \mathbb{R}^m$, and if $S_w \in \mathcal{M}_m$ is the transpose of*

$$S_w^T = \begin{bmatrix} A & \vdots & \mathbf{b} \\ \ldots & \ldots & \ldots \\ \mathbf{0}^T & \vdots & 1 \end{bmatrix}, \tag{9}$$

then S and S_w, and therefore S, A and $\mathbf{b}$, are connected by the relation

$$S = S_p\, S_w\, S_l = S_p\, S_w\, (S_p)^{-1} \tag{10}$$

where S_p and S_l are given by (5). Furthermore, if the spectrum of the A is $\sigma(A) = \{\lambda_i\}_{i=1}^{m-1}$, then the spectrum of S is $\sigma(S) = \{\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, 1\}$.

Proof. Let $\mathcal{L}_w \in \Lambda$ be the linear mapping associated to S_w, clearly

$$\mathcal{L}_w(\mathbb{R}_\perp^{m-1}) = \mathbb{R}_\perp^{m-1}.$$

More precisely $\mathcal{L}_w([\mathbf{x}^T \,|\, 1]^T) = S_w^T \, [\mathbf{x}^T \,|\, 1]^T = [w(\mathbf{x})^T \,|\, 1]^T$. Therefore

$$\mathcal{L}_w(i(\mathbf{x})) = \mathcal{L}_w([\mathbf{x}^T \,|\, 1]^T) = S_w^T \, [\mathbf{x}^T \,|\, 1]^T = [w(\mathbf{x})^T \,|\, 1]^T = i(w(\mathbf{x}))\,, \tag{11}$$

$\forall\, \mathbf{x} \in \mathbb{R}^{m-1}$. Consider, now, $S \in \mathcal{S}_m$ and the associated $\mathcal{L} \in \Lambda_S$, mapping $\mathbf{x} \in \mathbf{V}$ into $\mathcal{L}(\mathbf{x}) = S^T\mathbf{x} \in \mathbf{V}$. By Definition 6, $w = \Pi(\mathcal{L})$ if and and only if (8) holds. Since, by definition, $\pi(\mathbf{x}) = i^{-1}(\mathcal{L}_p(\mathbf{x}))\ \forall\, \mathbf{x} \in \mathbf{V}$, (8) is equivalent to

$$\mathcal{L}_p(\mathcal{L}(\mathbf{x})) = i\,(w\,(i^{-1}(\mathcal{L}_p(\mathbf{x}))))\,,$$

and , applying (11) for $\mathbf{x} = i^{-1}(\mathcal{L}_p(\mathbf{x})) \in \mathbb{R}^{m-1}$, this yields that $w = \Pi(\mathcal{L})$ iff

$$\mathcal{L}_p(\mathcal{L}(\mathbf{x})) = \mathcal{L}_w(i\,(i^{-1}\mathcal{L}_p(\mathbf{x}))) = \mathcal{L}_w(\mathcal{L}_p(\mathbf{x}))\,, \quad \forall\, \mathbf{x} \in \mathbf{V}. \tag{12}$$

Writing (12) in matrix form yields $S_p^T S^T \mathbf{x} = S_w^T S_p^T \mathbf{x}\,,\ \forall\, \mathbf{x} \in \mathbf{V}$ which leads to

$$S^T = (S_p^T)^{-1} S_w^T S_p^T$$

or, equivalently, (10).

Clearly, because of its particular block triangular structure, S_w has the same eigenvalues as A plus the eigenvalue $\lambda_m = 1$, namely $\sigma(S_w) = \{1\} \cup \sigma(A)$. But, by (10), S is similar to S_w , therefore $\sigma(S) = \sigma(S_w) = \{1\} \cup \sigma(A)\,.$ ∎

Remark 5 *We observe explicitly that by Theorem 1 the operator* Π *is invertible. The inverse* Π^{-1} *is responsible for adding to the spectrum of the transformation* w *an eigenvalue equal to one but, as far as the restriction of* $\mathcal{L} = \Pi^{-1}(w)$ *to* $\mathbf{V}$ *is concerned, this has no practical effect, by the orthogonality of* $\mathbf{V}$ *with the corresponding eigenvector of* S^T *(see Remark 3).*

Definition 7 *The IFS* $\Sigma = \{\mathbb{R}^{m-1}; w_1, \ldots, w_n\}$ *is called the projection of the canonical AIFS* $\Omega = \{\mathbf{E}; S_1, \ldots, S_n\,,\}$, *and it is written* $\Sigma = \Pi(\Omega)$, *if and only if* $w_i = \Pi(\mathcal{L}_i)$, $\forall\, i = 1, \ldots, n$, *where* $\mathcal{L}_i \in \Lambda$ *is the linear map associated to* S_i.

Corollary 1 *Given the IFS* $\Sigma = \{\mathbb{R}^{m-1}; w_1, \ldots, w_n\}$, *there is one and only one canonical AIFS* $\Omega = \{\mathbf{E}; S_1, \ldots, S_n\}$ *in* $\mathbb{R}^m$ *such that* $\Sigma = \Pi(\Omega)$, *and* $att(\Sigma) = \pi(att(\Omega))$ *holds.*

Proof. It is straightforward from Definition 7 and Theorem 1 ∎

Theorem 2 *Let a canonical AIFS* Ω *be given by* (4) *and let the IFS* Σ *be its projection. Then, if one of them is hyperbolic, the other one is hyperbolic too.*

Proof. Suppose that Ω be hyperbolic. Let $\mathcal{L} \in \Lambda_S$ be the linear mapping associated with some $S \in \Omega$. Being this a contraction in $\mathbf{V}$, there exist a vector norm $\|\cdot\|$ and a real number $0 < \alpha < 1$ such that, being $\mathbf{v}$ the vector $\mathbf{v} = \mathbf{x} - \mathbf{y}$,

$$\|\mathcal{L}(\mathbf{v})\| = \|S^T \mathbf{v}\| \leq \alpha \|\mathbf{v}\|, \quad \forall\ \mathbf{x},\, \mathbf{y} \in \mathbf{V}. \tag{13}$$

Let, now, $w = \Pi(\mathcal{L})$ with $w : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, and let $\rho(A) = max_{i=1}^{m-1} |\lambda_i|$ be the spectral radius of A. Since, by Theorem 1, $\sigma(A) \subset \sigma(S)$, for any $i \in \{1, \ldots, m-1\}$, $\lambda_i \in \sigma(A) \Rightarrow \lambda_i \in \sigma(S)$ and, denoting by $\mathbf{v}_i$ the corresponding right eigenvector of S^T, by Remark 2 and (13) we deduce

$$|\lambda_i| \|\mathbf{v}_i\| = \|\lambda_i \mathbf{v}_i\| = \|S^T \mathbf{v}_i\| \leq \alpha \|\mathbf{v}_i\|, \quad i = 1, \ldots, m-1.$$

This yields $|\lambda_i| \leq \alpha < 1$, $i = 1, \ldots, m-1$, and consequently $\rho(A) < 1$. Now, it is known that, given any matrix M, if the spectral radius of M is strictly smaller than one, there exists a natural matrix norm $\|\cdot\|_M$ such that $\|M\|_M < 1$ (see, e.g. [11] or [3]). Thus, denoting by $\|\cdot\|$ both a matrix norm such that $\|A\| < 1$ and a vector norm compatible with it, we have, $\forall\, \mathbf{x},\, \mathbf{y} \in \mathbb{R}^{m-1}$,

$$\|w(\mathbf{x}) - w(\mathbf{y})\| = \|A(\mathbf{x} - \mathbf{y})\| \leq \|A\| \, \|(\mathbf{x} - \mathbf{y})\| < \|\mathbf{x} - \mathbf{y}\|,$$

according to which w is a contraction in $\mathbb{R}^{m-1}$.

Vice versa, supposing that Σ be a hyperbolic IFS, consider any contraction $w : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ in Σ and let $\mathcal{L} \in \Lambda_S$ be the transformation in Ω such that $w = \Pi(\mathcal{L})$. Let $\mathbf{x} \neq \mathbf{y} \in \mathbf{V}$ and let $\{\mathbf{v}_i\}_{i=1}^{m-1}$ be the right eigenvectors of S^T corresponding to the non-unit eigenvalues, then (see Remark 3) there exist real constants $\{\alpha_i\}_{i=1}^{m-1}$ such that the vector $(\mathbf{x} - \mathbf{y}) = \sum_{i=1}^{m-1} \alpha_i \mathbf{v}_i$ and therefore

$$\|\mathcal{L}(\mathbf{x}) - \mathcal{L}(\mathbf{y})\| = \|S^T(\mathbf{x} - \mathbf{y})\| = \left\| S^T \sum_{i=1}^{m-1} \alpha_i \mathbf{v}_i \right\| = \left\| \sum_{i=1}^{m-1} \alpha_i S^T \mathbf{v}_i \right\| = \left\| \sum_{i=1}^{m-1} \alpha_i \lambda_i \mathbf{v}_i \right\|$$

$$\leq \left\| \sum_{i=1}^{m-1} \alpha_i \rho(A) \mathbf{v}_i \right\| = \rho(A) \left\| \sum_{i=1}^{m-1} \alpha_i \mathbf{v}_i \right\| = \rho(A)\, \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|.$$

By the arbitrariness of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{V}$, $\mathcal{L}$ is a contraction in $\mathbf{V}$. ■

3 Cantor set and Cantor dust

Probably, the first fractal set in history was published by Georg Cantor in 1883 [2], much earlier than the term *fractal* was ever coined. Not widely accepted by mathematicians at the time, the Cantor set $\mathbf{C}$ gained popularity a hundred years later, with the development of the new theory of fractal sets. The essence of the definition given in [2] is as follows.

Definition 8 (Cantor set: set theoretic definition)

Step 1. $C_0 = [0, 1]$;

Step 2. For $n \in \mathbb{N} : C_n = C_{n-1} \setminus \{\text{middle third of all intervals in } C_{n-1}\}$;

Step 3. $\mathbf{C} = \cap_{n \in \mathbb{N}_0} C_n$.

Since its very birth this *Cantor "middle third" set* seemed to be a puzzle, enjoying different properties, some of which were (only apparently) contradictory.

Proposition 1 (Properties of the Cantor set) P1. $\mathbf{C}$ *is a nonempty compact subset of* $\mathbb{R}$ *;* P2. $\mathbf{C} \subset [0, 1]$ *;* P3. $\mathbf{C}$ *is a perfect set ;* P4. $\mathbf{C}$ *is nowhere dense in* $\mathbb{R}$ *;* P5. $\mathbf{C}$ *is of zero length and does not contain any open set ;* P6. $\mathbf{C}$ *is totally disconnected ;* P7. $\mathbf{C}$ *has zero topological dimension.*

At first sight, it may sound surprising that $\mathbf{C}$ be perfect but nowhere dense in $\mathbb{R}$, yet this very property deserved to this set the alternative name of *Cantor discontinuum.* All the points in $\mathbf{C}$ are frontier points, namely $\mathbf{C} = \partial\mathbf{C}$, but all the points in $\mathbf{C}$ are also *accumulation points*, as $\mathbf{C} = \overline{\mathbf{C}}$. The fact that $\mathbf{C}$ is uncountably infinite but has null length also sounds paradoxical. But all these peculiarities and apparent discrepancies can be explained by the distinction of the topological dimension of $\mathbf{C}$, $D_T(\mathbf{C})$, from its Hausdorff dimension $D_H(\mathbf{C})$. In fact, as we said, $D_T(\mathbf{C}) = 0$, while $D_T(\mathbf{C}) \neq D_H(\mathbf{C}) = \log 2/\log 3 = 0.63092$.

It was shown by Mandelbrot, in [10], that the property of $\mathbf{C}$ of having zero topological dimension is a consequence of being totally disconnected. He also noted that there is a large family of sets that share this characteristic with $\mathbf{C}$, and coined for them the new term *dust*, giving the following definition:

Definition 9 (Dust) *A set* $\mathbf{X}$ *such that* $D_T(\mathbf{X}) = 0$ *is called a dust.*

Borrowing Mandelbrot's terminology, we call *dust* a totally disconnected subset of $\mathbb{R}^m$, for any m. The AIFS theory developed in the previous sections offers us an easy tool to construct and represent dusts in spaces of arbitrary dimension.

It is known that, in the context of IFS theory, the Cantor set $\mathbf{C}$ can be introduced as the attractor of a two term affine IFS (see [1], [10])

Definition 10 (Cantor set: definition by IFS) *The Cantor set* $\mathbf{C}$ *is the attractor of the IFS* $\{\,[0,1]\,;\ w_1, w_2\}$ *with* $w_1(x) = x/3, \quad w_2(x) = (x+2)/3$.

In the AIFS theory context, by Theorem 1 and Corollary 1 in Section 2, the Cantor set can be introduced as the attractor of an AIFS in $\mathbb{R}^2$. In fact, denoting by $\mathbf{E}^{\{m\}}$ the set of vertices of the canonical m-simplex (for any $m \geq 2$),

Definition 11 (Cantor set: definition by AIFS) *The* Cantor set $\mathbf{C}$ *is the* π*-projection of the attractor of the AIFS* $\Xi = \{\,\mathbf{E}^{\{2\}}\,;\, S_1,\, S_2\}$ *where*

$$S_1 = \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1/3 & 2/3 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

Furthermore, the particular nature of the AIFS model makes generalization of the previous definition to m-dimensional spaces straightforward.

Definition 12 (Multi-dimensional Cantor dust: definition by AIFS) *For any* $m \geq 2$ *and* $0 < p < 1/2$*, let the matrices* $S_k = [s^k_{ij}(p)]_{i,j}$ *have the elements*

$$s^k_{ij}(p) = \begin{cases} (1-p) + p\,\delta_{ij} & \text{if } j = k, \\ p\,\delta_{ij} & \text{if } j \neq k, \end{cases} \qquad i, j = 1, 2, ..., m \tag{15}$$

for $k = 1, ..., m$. The attractor of the AIFS

$$\Xi_{(m,\,p)} \;=\; \{\mathbf{E}^{\{m\}}\,;\, S_1(p),\, ...,\, S_m(p)\,\},$$

denoted by $\mathbf{C}^{(m,p)}$, is called an $(m-1)$-dimensional (generalized) Cantor dust.

Remark 6 *Definition* 12 *yields the middle third Cantor set for $m = 2$, $p = 1/3$. Namely $\mathbf{C} = \pi(\mathbf{C}^{(2,\,1/3)})$.*

The generalized Cantor dust $\mathbf{C}^{(m,\,p)}$ is a self-similar fractal, and its Hausdorff dimension can be calculated by the formula for box-dimension [1],

$$D_H(\mathbf{C}^{(m,p)}) = \frac{\log m}{\log(1/p)}. \tag{16}$$

If we consider the dusts obtained by keeping the dimension $\widehat{m}$ fixed and allowing for all feasible values of p, we obtain the family $\mathcal{D}_{\widehat{m}} = \{\mathbf{C}^{(\widehat{m},\,p)}\}_{p\in(0,1/2)}$. We see from (16) that the Hausdorff dimension $D_H(\mathbf{C}^{(\widehat{m},\,p)})$ increases with p, ranging from 0 to $(\log\widehat{m}/\log 2)$ as p ranges from 0 to 0.5.

A different behavior is exhibited by the family $\mathcal{D} = \{\mathbf{C}^{(m,\,1/m)}\}_{m\geq 3}$ since, by (16), all members of $\mathcal{D}$ have Hausdorff dimension 1. In particular, the Cantor dust $\mathbf{C}^{(3,\,1/3)} \in \mathcal{D}$ is a subset of $\mathbb{R}^2$. It is a different set, though, from the Cartesian auto-product of the Cantor set ($\mathbf{C}^2 = \mathbf{C}\times\mathbf{C}$) discussed in [4]. By its nature, $\mathbf{C}^{(3,\,1/3)}$ is closer to the Sierpinski triangle than to $\mathbf{C}^2$. In fact, the Sierpinski triangle can be obtained as $\mathbf{C}^{(3,\,1/2)}$, but it must be remarked that, according to Definition 12, this is not a dust any more, since $p = 1/2$. This is consistent with the well known fact that the Sierpinsky triangle $\mathbf{S}_\triangle$ is a connected subset of $\mathbb{R}^2$, actually a *plane open curve* whence $D_T(\mathbf{S}_\triangle) = 1$, while $D_H(\mathbf{S}_\triangle) = log3/log2$.

4 Totally disconnected AIFS

As we mentioned, the Cantor set $\mathbf{C}$ is characterized by the property of being a totally disconnected set (P6). In [1], Barnsley defines the totally disconnected IFS. An analogous definition can be given for AIFS, namely

Definition 13 *The hyperbolic AIFS $\Omega = \{\mathbf{E}^{\{m\}}; S_1, \ldots, S_n\}$ with regular matrices S_i and with the attractor A is totally disconnected if*

$$\mathcal{L}_i(A) \cap \mathcal{L}_j(A) = \varnothing, \text{ all } i \neq j, \tag{17}$$

where $\mathcal{L}_i$ is the linear mapping associated with the matrix S_i, $i = 1, \ldots, n$.

In order to characterize attractors of totally disconnected AIFS, we need to preliminarily introduce a new property, the *convex hull property.*

For the Cantor set $\mathbf{C}$, property P2 in Proposition 1 states that it is contained in $[0,1]$ namely in the the π-projection of the canonical 2-simplex. By Remark

6 this is equivalent to $\mathbf{C}^{(2,1/3)} \subset \mathrm{conv}\{\mathbf{E}^{\{2\}}\} = \Delta_{\mathbf{E}}$. It is natural to wonder if such an inclusion property is valid for arbitrary m and p. Actually it is, as stated by Corollary 2, below. This is straightforward, since, as was proved in [6],

Theorem 3 *If all the matrices in the AIFS* $\Omega = \{\mathbf{E}^{\{m\}}; S_1, ..., S_n\}$ *are non-negative, then* Ω *has the* convex hull property, *namely* $\mathrm{att}(\Omega) \subset \mathrm{conv}\{\mathbf{E}^{\{m\}}\}$.

Corollary 2 *The AIFS* $\Xi_{(m,\,p)}$ *in* Definition 12 *has the convex hull property. Therefore, for any* $m \geq 2$ *and any* $p \in (0, 1/2)$, $\mathbf{C}^{(m,p)} \subset \mathrm{conv}\{\mathbf{E}^{\{m\}}\}$.

Proof. If $p \in (0, 1/2)$, according to (15) $0 < s^k_{i,j} < 1$ for all $i, j, k = 1, \ldots, m$ and for any arbitrary m. Therefore the assertion follows from Theorem 3 . ∎

Furthermore, for any given hyperbolic AIFS, there always exists another hyperbolic AIFS with the same attractor and with the convex hull property.

Theorem 4 *For any hyperbolic canonical AIFS* $\Omega = \{\mathbf{E}^{\{m\}}; S_1, ..., S_n\}$, *there exists an AIFS* $\Omega_{\mathbf{U}} = \{\mathbf{U}; P_1, \ldots, P_n\}$ *that satisfies*

(i) $\mathrm{att}(\Omega_{\mathbf{U}}) = \mathrm{att}(\Omega)$;

(ii) $\Omega_{\mathbf{U}}$ *has the convex hull property, i.e.* $\mathrm{att}(\Omega_{\mathbf{U}}) \subset \mathbf{U}$.

Proof. Let $W_\Omega = \cup_i \mathcal{L}_i$ be the Hutchinson operator associated with Ω, and let $\mathbf{E} = \mathbf{E}^{\{m\}}$, $\Delta_{\mathbf{E}} = \mathrm{conv}\{\mathbf{E}\}$. Obviously, the set $B = \cup_{k=0}^{+\infty} W^k_\Omega(\Delta_{\mathbf{E}}) \supset \mathrm{att}(\Omega)$ is a bounded subset of $\mathbf{V}$. Then, there exists in $\mathbf{V}$ an m-simplex $\mathbf{U}$ such that $B \subset \mathrm{conv}(\mathbf{U}) = \Delta_{\mathbf{U}}$, which implies $\Delta_{\mathbf{E}} \subset \Delta_{\mathbf{U}}$. But, then, also $\mathbf{E} \subset \Delta_{\mathbf{U}}$ which implies the existence of a non-negative row-stochastic matrix M such that $\mathbf{E} = M\,\mathbf{U}$ and therefore also $S_i\mathbf{E} = S_i M\mathbf{U} \subset \Delta_{\mathbf{U}}$, for $i = 1, \ldots, n$. Let $P_i = S_i M$ and $\Omega_{\mathbf{U}} = \{\mathbf{U}; P_1, \ldots, P_n\}$. Note that the orbits $\{W^k_{\mathbf{U}}(\Delta_{\mathbf{U}})\}_{k=1}^{+\infty}$ and $\{W^k_\Omega(\Delta_{\mathbf{E}})\}_{k=1}^{+\infty}$ are identical, which implies the inclusion $W^k_{\mathbf{U}}(\Delta_{\mathbf{U}}) \subset \Delta_{\mathbf{U}}$, $\forall\, k \in \mathbf{N}$. And, because of this, $\lim_k W^k_{\mathbf{U}}(\Delta_{\mathbf{U}}) = \mathrm{att}(\Omega_{\mathbf{U}}) = \mathrm{att}(\Omega) \subset \Delta_{\mathbf{U}}$ ∎

Lemma 1 *Let* $\Delta_{\mathbf{E}} = \mathrm{conv}(\mathbf{E}^{\{m\}})$. *If the AIFS* $\Omega = \{\mathbf{E}^{\{m\}}; S_1, \ldots, S_n\}$ *has convex hull property and it satisfies*

$$\mathcal{L}_i(\Delta_{\mathbf{E}}) \cap \mathcal{L}_j(\Delta_{\mathbf{E}}) = \varnothing, \text{ all } i \neq j, \tag{18}$$

then it is totally disconnected.

Proof. $A \subset \Delta_{\mathbf{E}} \Rightarrow \mathcal{L}_i(A) \subset \mathcal{L}_i(\Delta_{\mathbf{E}})$. So if the sets $\left\{\mathcal{L}_i(\Delta_{\mathbf{E}})\right\}_{i=1}^n$ are mutually disjoint, the sets $\{\mathcal{L}_i(A)\}_{i=1}^n$ are, too. And (17) holds. ∎

Theorem 5 *If the hyperbolic AIFS* $\Omega = \{\mathbf{E}^{\{m\}}; S_1, \ldots, S_n\}$ *satisfies the hypotheses of Lemma 1, the attractor of* Ω *is a totally disconnected set.*

Proof. For fixed m, let $\mathbf{E} = \mathbf{E}^{\{m\}}$ and let $\Omega = \{\mathbf{E}; S_1, \ldots, S_n\}$ have the convex hull property, so that $A = \mathrm{att}(\Omega) \subset \Delta_{\mathbf{E}}$. Let $W_\Omega(\cdot) = \cup_i \mathcal{L}_i(\cdot)$ be the Hutchinson operator associated with Ω, and let (18) hold. We will prove by contradiction that, under such conditions, A is a totally disconnected set. By definition, a set is totally disconnected iff it does not contain any non-empty connected set that is not a single point. So, let $K(\rho)$ be a closed ball having radius $\rho > 0$ such that $K(\rho) \subset A$. We will prove that $\rho > 0$ is false.

Let $\Delta_0 = \Delta_{\mathbf{E}}$ and $\Delta_k = W_\Omega^k(\Delta_{\mathbf{E}})$, for $k \in \mathbf{N}$. Note that, by (18), Δ_k is the union of n^k disjoint sub-simplices of the simplex $\Delta_{\mathbf{E}}$, namely the sets $\Delta_{j_1,\ldots,j_k} = \mathcal{L}_{j_k}(\ldots(\mathcal{L}_{j_2}(\mathcal{L}_{j_1}(\Delta_{\mathbf{E}}))))$ for all possible combinations of k indices such that $j_i \in \{1, \ldots n\}$, $\forall\, i$. Also notice that, since every $\mathcal{L}_j$ is a contraction, with the factor s_j, denoting by $s = \max_j s_j < 1$, and by $D(X)$ the diameter of the set X, it holds that $D(\Delta_{j_1,\ldots,j_k}) < s^k D(\Delta_{\mathbf{E}}) = d(k)$, for every k and for every set of indices. Therefore, for any fixed $\overline{\rho}$ there exists k_ρ such that, if $k > k_\rho$, $D(\Delta_{j_1,\ldots,j_k}) < \overline{\rho}$ for any set of indices.

Suppose, now, that there exists a closed ball $K(\rho)$, $\rho > 0$ such that $K(\rho) \subset A$. Then, $\forall\, k \in \mathbf{N}$, it must be $K(\rho) \subset A = \cap_{h=0}^{\infty} \Delta_h \subset \Delta_k$. And, since $K(\rho)$ is obviously connected, it must be entirely contained in one of the connected parts of Δ_k. In other words, for every k there must be one and only one set of indices such that $K(\rho) \subset \Delta_{j_1,\ldots,j_k}$. Which implies $\rho < D(\Delta_{j_1,\ldots,j_k}) < d(k)$. By the arbitrariness of k, this would imply $\rho = 0$, contradicting the hypothesis that $\rho > 0$. Thus, $A = \mathrm{att}(\Omega)$ cannot contain a closed ball of positive radius i.e., it is a totally disconnected set. ∎

Finally we state a condition on the matrices in in the AIFS Ω under which Ω is totally disconnected, and $\mathrm{att}(\Omega)$ is a totally disconnected set, or *a dust.* Let $\mathrm{sgn}\{\mathbf{v}\} = +1$ if all the components of $\mathbf{v} \in \mathbb{R}^m$ are positive, and $\mathrm{sgn}\{\mathbf{v}\} = -1$ if they are negative.

Theorem 6 *If the AIFS $\Omega = \{\mathbf{E}^{\{m\}}; S_1, \ldots, S_n\}$ has the convex hull property and if, for $1 \le i < j \le n$, there exist vectors $\mathbf{a}_{ij}$ such that*

$$\mathrm{sgn}\{S_i\, \mathbf{a}_{ij}\} = -\,\mathrm{sgn}\{S_j\, \mathbf{a}_{ij}\},\ 1 \le i < j \le n, \tag{19}$$

then Ω is totally disconnected.

Proof. Let $\mathbf{E} = \mathbf{E}^{\{m\}}$, and $\Delta_i = \mathcal{L}_i(\Delta_{\mathbf{E}})$, $\forall\, i \in \{1, \ldots, n\}$. By Lemma 1, Ω will be totally disconnected if $\forall\, i \ne j$, $\Delta_i \cap \Delta_j = \varnothing$. Now, since Δ_i and Δ_j are convex subsets of $\mathbb{R}^m$, if they have empty intersection there must be some hyperplane separating them. Recall that, in $\mathbb{R}^m$, given a vector $\mathbf{a} = [a_1\, \ldots\, a_m]^T$, the equation

$$\mathbf{H}(\mathbf{r}) = \langle \mathbf{a}, \mathbf{r} \rangle = \sum_{j=1}^{m} a_j \rho_j = 0, \tag{20}$$

represents the hyperplane $\mathbf{H}_\mathbf{a}$ of dimension $m-1$, passing through the origin and orthogonal to the vector $\mathbf{a}$. This hyperplane splits the space $\mathbb{R}^m$ to two half-spaces, $\mathbf{H}(\mathbf{r}) < 0$ and $\mathbf{H}(\mathbf{r}) > 0$. Suppose $\mathbf{x}$ and $\mathbf{y}$ are position vectors of the corresponding points from $\mathbf{V}$ such that they belongs to different half-spaces. Then, signs of scalar products $\mathbf{x}^T\mathbf{a} = \mathbf{H}(\mathbf{x})$ and $\mathbf{y}^T\mathbf{a} = \mathbf{H}(\mathbf{y})$ are different and vice versa.

Note that the rows of matrix S_i represents vertices of the simplex Δ_i. If (19) is valid then there exists a hyperplane $\mathbf{H}_{\mathbf{a}_{ij}}$ of the form (20) with orthogonal vector $\mathbf{a}_{ij}$ that separates simplices Δ_i and Δ_j. The rest follows from Lemma 1. ∎

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