From twists to involution bands

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Abstract

In this note[1] we give a representation theorem for involution bands. In addition, it is shown that the construction used in such a representation yields an operator on the lattice of involution band varieties.

§1. By an *involution semigroup* we mean a semigroup S endowed with a unary operation * (called the *involution*) such that the identities $(xy)^* = y^*x^*$ and $(x^*)^* = x$ are satisfied. Groups and, more generally, inverse semigroups are clearly the most natural examples of involution semigroups. The intensive study of involution semigroups began with the paper of Nordahl and Scheiblich [15], where involution semigroups with the additional identity $xx^*x = x$, called *regular *-semigroups* (or just **-semigroups* for short), were considered.

Idempotent involution semigroups, that is, *involution bands*, constitute a separate topic within the involution semigroup theory. Following the description of the lattice of all varieties of bands [2, 9, 10, 11] (see also [20, 22]), Adair [1] provided a complete description of the lattice of all *-band varieties. A somewhat broader lattice of varieties of involution bands was provided by the author [6], while all varieties of normal involution bands were described in [7]. See also [8] for some related results.

In studying the structure of bands, spined (pullback) products turned out to be one of the most powerful tools. First Kimura [13] proved that any regular band S (a band satisfying $xyxzx = xyzx$) is a spined product of a left regular band ($xyx = xy$) and a right regular band ($xyx = yx$) with respect to the structure semilattice $S/\mathcal{D}$ of S. Later, this was generalized (along with some results of Petrich [17, 18, 19]) by Ćirić and Bogdanović [5] who proved that if $\mathcal{U}$ and $\mathcal{V}$ are arbitrary homotypical varieties of bands then actually any band from $\mathcal{U} \vee \mathcal{V}$ is a spined product of a band from $\mathcal{U}$ and a band from $\mathcal{V}$ with respect to a member of $\mathcal{U} \wedge \mathcal{V}$.

Concerning involution bands, Scheiblich [21] proved the *-band analogue of Kimura's result. Namely, S is a *-band which is also a regular band if and only if it is a spined product of a left regular band and its dual band (which is a right regular band) with respect to a semilattice, where the involution is just the operation of reversing pairs. Our main objective in this note is to extend

[1] Presented at the IMC "Filomat 2001", Niš, August 26–30, 2001

2000 Mathematics Subject Classification: 20M07, 20M10

Keywords: Involution semigroup, involution band, regular semigroup, twisting operator

this result to all involution bands by introducing the notion of a *twisted spined square* of a band. This construction will provide us with an operator $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{tw}}$ on the lattice of varieties of involution bands, called the *twisting operator*, which certainly contains some information on the structure of the considered lattice.

§2. We begin with some definitions and notations. For any semigroup S (whose operation is denoted by juxtaposition), the dual semigroup of S (the semigroup with universe S, whose operation is given by $a \diamond b = ba$) is denoted by S^{∂}. A semigroup variety $\mathcal{V}$ is *central* if $S \in \mathcal{V}$ implies $S^{\partial} \in \mathcal{V}$ for any S. Centrality is easily seen to be equivalent with the following syntactic property: for any identity $u = v$ that holds on $\mathcal{V}$, the identity $\overline{u} = \overline{v}$ also holds in $\mathcal{V}$, where $\overline{w}$ denotes the word obtained by reversing w. It is easy to extract central varieties of bands from the results of [2, 9, 10]: they are depicted in the following figure (which, in addition, defines a sequence $\mathcal{B}_i$,
$leqi \leq \infty$, of central homotypical band varieties).

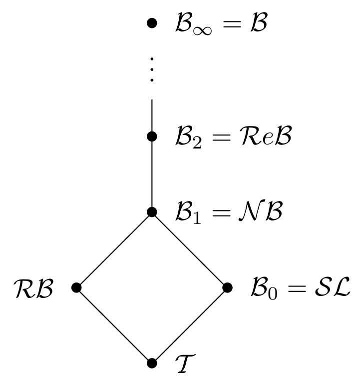

Figure 1. All central varieties of bands

If $\mathcal{V}$ is a variety of involution bands, let $\underline{\mathcal{V}}$ denote the band variety generated by semigroup reducts of all members of $\mathcal{V}$. Equivalently, we are concerned with the band variety determined by the *-free identities of $\mathcal{V}$. Obviously, $\underline{\mathcal{V}}$ must be a central variety. We define the *type* of $\mathcal{V}$, $t(\mathcal{V})$, such that $t(\mathcal{V}) = i$ if and only if $\underline{\mathcal{V}} = \mathcal{B}_i$, while for the other cases we set $t(\mathcal{V}) = \underline{\mathcal{V}}$.

By the main result of [1], the varieties of *-bands are in a one-to-one correspodence with the central band varieties. Thus, for any central variety of bands $\mathcal{V}$ we have one variety of *-bands of the corresponding type, which we denote by $\mathcal{V}^{\mathrm{reg}}$. Similarly, we define $\mathcal{V}^*$ as the variety of *all* involution bands whose semigroup reducts belong to $\mathcal{V}$ (this is in fact the greatest variety of the type which corresponds to $\mathcal{V}$). In particular, $\mathcal{R}\mathcal{B}^{\mathrm{reg}} = \mathcal{R}\mathcal{B}^*$. The subvariety of $\mathcal{V}^*$ defined by the identities $xx^*y = xyx^* = xx^*$ (whose effect is precisely that any product containing a factor a and its star a^* equals to zero) is denoted by $\mathcal{V}^0$. Varieties of this form were investigated in more detail in [4].

Finally, for a semigroup variety $\mathcal{V}$, let $\mathcal{V}^\partial$ denote its dual variety, consisting of duals of members of $\mathcal{V}$. Note that for band varieties, the conditions $\mathcal{V} \vee \mathcal{V}^\partial = \mathcal{B}_{i+2}$ and $\mathcal{V} \wedge \mathcal{V}^\partial = \mathcal{B}_i$ for some $i \geq 0$ determine $\mathcal{V}$ up to duality. Among the two possible 'solutions' we are going to choose one of them, $\mathcal{LB}_{i+2}$, such that $\mathcal{LB}_2$ is the variety of left regular bands and that for all $j \geq 2$ we have $\mathcal{LB}_j \leq \mathcal{LB}_{j+1}$.

§3. We now turn to some special congruences on (involution) bands. As known, for every equivalence relation ρ on a semigroup S there exists the greatest congruence relation contained in ρ, which is called the *congruence opening of* ρ and denoted by $\rho^\flat$. We have $(a,b) \in \rho^\flat$ if and only if $(xay, xby) \in \rho$ for all $x, y \in S^1$.

In general, the well-known Green's equivalences $\mathcal{L}$ and $\mathcal{R}$ are not congruences, even if we restrict ourselves to bands (however, the equivalence $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ is equal to the least semilattice congruence on a band). Therefore, it is meaningful to consider $\mathcal{L}^\flat$ and $\mathcal{R}^\flat$. In particular, bearing in mind the simple form that $\mathcal{L}$ and $\mathcal{R}$ take on bands and the fact that $\mathcal{R}$ (resp. $\mathcal{L}$) is a left (right) congruence on any semigroup, we obtain that $(a,b) \in \mathcal{R}^\flat$ if and only if $ax = bxax$ and $bx = axbx$ for all $x \in S$ (while $\mathcal{L}^\flat$ is defined by the dual condition).

In [5], Ćirić and Bogdanović proved that (just as $\mathcal{L}$ and $\mathcal{R}$ do in an arbitrary semigroup), $\mathcal{L}^\flat$ and $\mathcal{R}^\flat$ commute in any band, so that we obtain their join $\mathcal{D}'$ simply as their composition $\mathcal{L}^\flat \circ \mathcal{R}^\flat$. Also, a number of descriptions of $\mathcal{D}'$ are given in [5], from which we single out the following one.

Proposition 1 ([5, Proposition 1]) *Let S be any band and $a, b \in S$. Then $(a,b) \in \mathcal{D}'$ if and only if for any $x, y \in S$ we have $xay = xaxbay$ and $xby = xbayby$.*

Sometimes, we shall use lower indices such as $\mathcal{R}^\flat_S$ to stress that we consider the corresponding relation on S. For convenience, the relations introduced so far are depicted in the following diagram (it is clear that on a band S we have $\mathcal{L}_S \cap \mathcal{R}_S = \mathcal{L}^\flat_S \cap \mathcal{R}^\flat_S = \Delta_S$).

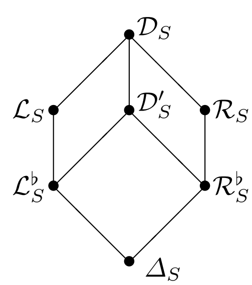

Figure 2. Green's relations and some related congruences

Now let S be an involution band. Of course, we can consider S as an ordinary band too, and deal with $\mathcal{L}^\flat_S$ and $\mathcal{R}^\flat_S$ in that sense, as we are going to do in the rest the paper (it is easy to see that the identity relation on S is the greatest *-congruence contained in any of $\mathcal{R}$ and $\mathcal{L}$). Also, this is the way in which quotients $S/\mathcal{R}^\flat$ and $S/\mathcal{L}^\flat$ should be understood.

If ρ is any relation on S, we define its *dual relation* ρ^* by $(a,b) \in \rho^*$ if and only if $(a^*, b^*) \in \rho$. In involution bands, we have the following symmetry.

Lemma 2 *In any involution band S, $\mathcal{R} = \mathcal{L}^*$ and $\mathcal{R}^\flat = (\mathcal{L}^\flat)^*$.*

Proof. Obviously, we have $(a,b) \in \mathcal{R}$ if and only if $ab = b$ and $ba = a$. By applying involution, the latter condition is equivalent to $b^*a^* = b^*$ and $a^*b^* = a^*$, i.e. to $(a^*, b^*) \in \mathcal{L}$, $(a,b) \in \mathcal{L}^*$. As for any equivalence ρ of S we easily have $(\rho^*)^\flat = (\rho^\flat)^*$, the second assertion of the lemma also follows. ∎

Hence, for each element a of an involution band S, we have $a\mathcal{R}^\flat = a^*\mathcal{L}^\flat$. Also, the following important observation is worth noting.

Lemma 3 *In any involution band S, $\mathcal{D}'$ is a *-congruence.*

Proof. This can be immediately seen from Proposition 1 above, or from the previous lemma and the fact that $\mathcal{D}'$ is obtained as $\mathcal{D}' = (\mathcal{R}^\flat)^* \circ \mathcal{R}^\flat = \mathcal{R}^\flat \circ (\mathcal{R}^\flat)^*$, which yields that $\mathcal{D}'$ is preserved by the involution. ∎

Therefore, whenever S is an involution band, the quotient $S/\mathcal{D}'$ will be equipped with a natural involution $(a\mathcal{D}')^* = a^*\mathcal{D}'$.

§4. Let S be an arbitrary band. Assume that there is an onto homomorphism $\xi : S \to T$, where T is an involution band. Then there is a natural way to define a homomorphism $\xi^\partial : S^\partial \to T$, namely

$$\xi^\partial(x) = \xi(x)^*.$$

Indeed, for every $x, y \in S$ we have

$$\xi^\partial(xy) = \xi(xy)^* = (\xi(x)\xi(y))^* = \xi(y)^*\xi(x)^* = \xi^\partial(x) \diamond \xi^\partial(y).$$

Since S and S^∂ share the same homomorphic image T, we can define their spined product with respect to T and ξ, $S \otimes_{T,\xi} S^\partial$. This is just the subsemigroup of the direct product $S \times S^\partial$ determined by

$$\bigcup_{a \in T} \left(\xi^{-1}(a) \times (\xi^\partial)^{-1}(a)\right).$$

In fact, we are concerned with a subdirect product of S and S^∂. Further, define a unary operation $^\sim$ on the above set by $(x,y)^\sim = (y,x)$. Of course, this operation is well defined since $\xi(y) = \xi^\partial(y)^* = \xi(x)^* = \xi^\partial(x)$. It is easy to verify that the above subdirect product equipped with $^\sim$ forms an involution band, which we call a *twisted spined square of S over T*.

The importance of this construction for involution bands is explained by the following result.

Theorem 4 *Every involution band S is isomorphic to a twisted spined square of $S/\mathcal{R}^\flat$ over $S/\mathcal{D}'$.*

Proof. First of all, we note that $S/\mathcal{L}^\flat \cong (S/\mathcal{R}^\flat)^\partial$. Namely, consider the mapping φ which maps $a\mathcal{L}^\flat$ to $a^*\mathcal{R}^\flat$. Then $a\mathcal{L}^\flat \cdot b\mathcal{L}^\flat = ab\mathcal{L}^\flat$ is mapped to $(ab)^*\mathcal{R}^\flat = b^*\mathcal{R}^\flat \cdot a^*\mathcal{R}^\flat = \varphi(a\mathcal{L}^\flat) \diamond \varphi(b\mathcal{L}^\flat)$, confirming that φ is a homomorphism. Since $(\mathcal{R}^\flat)^* = \mathcal{L}^\flat$, it is obviously a bijection.

Let $\nu : S/\mathcal{R}^\flat \to S/\mathcal{D}'$ be the natural homomorphism defined by $\nu(a\mathcal{R}^\flat) = a\mathcal{D}'$, $a \in S$. As for all $a \in S$ we have $a\mathcal{L}^\flat = a^*\mathcal{R}^\flat$, it follows

$$\nu^\partial(a\mathcal{L}^\flat) = \nu(a^*\mathcal{R}^\flat)^* = (a^*\mathcal{D}')^* = (a^*)^*\mathcal{D}' = a\mathcal{D}' = \nu(a\mathcal{R}^\flat).$$

Now define a function $\psi : S \to S/\mathcal{R}^\flat \otimes_{S/\mathcal{D}',\nu} S/\mathcal{L}^\flat$ by

$$\psi(a) = (a\mathcal{R}^\flat, a\mathcal{L}^\flat).$$

By the above remarks, this function is well defined. It remains to show that it is a *-homomorphism, since it is clearly a bijection. Indeed,

$$\begin{aligned} \psi(ab) &= (ab\mathcal{R}^\flat, ab\mathcal{L}^\flat) = (a\mathcal{R}^\flat \cdot b\mathcal{R}^\flat, a\mathcal{L}^\flat \cdot b\mathcal{L}^\flat) = \\ &= (a\mathcal{R}^\flat, a\mathcal{L}^\flat)(b\mathcal{R}^\flat, b\mathcal{L}^\flat) = \psi(a)\psi(b) \end{aligned}$$

and

$$\psi(a^*) = (a^*\mathcal{R}^\flat, a^*\mathcal{L}^\flat) = (a\mathcal{L}^\flat, a\mathcal{R}^\flat) = (a\mathcal{R}^\flat, a\mathcal{L}^\flat)^\sim = \psi(a)^\sim,$$

as required. ∎

The above decomposition of an involution band S we call the *standard twisted representation* of S. Our aim is now to 'position' the factors of this representation.

Lemma 5 *The following conditions are equivalent for a band (an involution band) S and an integer $i \geq 0$:*

(1) $S \in \mathcal{B}_{i+2}$ ($S \in \mathcal{B}^*_{i+2}$),

(2) $S/\mathcal{R}^\flat \in \mathcal{LB}_{i+2}$,

(3) $S/\mathcal{D}' \in \mathcal{B}_i$ ($S/\mathcal{D}' \in \mathcal{B}^*_i$).

Proof. (1)⇒(3) By Theorems 8 and 11 of [5], $S/\mathcal{D}' \in \mathcal{LB}_{i+2} \wedge \mathcal{LB}^\partial_{i+2} = \mathcal{B}_i$ (where $\mathcal{LB}_{i+2}$ is just $\mathbf{V}_{\mathcal{R}^0}$ in the notation of [5]).

(2)⇒(1) If $S/\mathcal{R}^\flat \in \mathcal{LB}_{i+2}$, then $S/\mathcal{L}^\flat \in \mathcal{LB}^\partial_{i+2}$, and so by Theorem 12 of [5] we have that $S \in \mathcal{LB}_{i+2} \vee \mathcal{LB}^\partial_{i+2} = \mathcal{B}_{i+2}$.

(3)⇒(2) By Corollary 1 of [5], $(S/\mathcal{R}^\flat_S)/\mathcal{L}^\flat_{S/\mathcal{R}^\flat_S}$ is a homomorphic image of $S/\mathcal{D}'$, and so it also belongs to $\mathcal{B}_i$. By Theorem 4 of [3] (or, alternatively, by Theorem 9 of [16]), we obtain that $S/\mathcal{R}^\flat_S$ belongs to the Mal'cev product $\mathcal{LZ} \circ \mathcal{B}_i = \mathcal{LB}_{i+2}$. ∎

The above lemma motivates us define $\mathcal{V}^{\mathrm{tw}}$ as the class of all involution bands S such that $S/\mathcal{D}' \in \mathcal{V}$, where $\mathcal{V} \neq \mathcal{RB}^*$ is some involution band variety (actually, we can omit this constraint by seeing that $(\mathcal{T}^*)^{\mathrm{tw}} = (\mathcal{RB}^*)^{\mathrm{tw}} = \mathcal{RB}^*$). Obviously, if $t(\mathcal{V}) = i \geq 0$, we have $t(\mathcal{V}^{\mathrm{tw}}) = i + 2$. It is not difficult to see that $\mathcal{V}^{\mathrm{tw}}$ is actually the class of all involution bands isomorphic to twisted spined squares of members of $\mathcal{LB}_{i+2}$ over members of $\mathcal{V}$. The principal virtue of the just introduced operator $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{tw}}$ is expressed by

Theorem 6 *$\mathcal{V}^{\mathrm{tw}}$ is a variety.*

Proof. Assume $\Theta = \{u_i = v_i : \ i \in I\}$ is a set of identities that defines $\mathcal{V}$ (within $\mathcal{B}^*$). Further, assume that all variables which appear in the identities from Θ are contained in the set $X = \{x_1, x_2, x_3, \ldots\}$, while y and z are two variables such that $y, z \notin X$. Consider the following set of identities:

$$\Theta^{\mathrm{tw}} = \{yu_iz = yu_iyv_iu_iz : \ i \in I\} \cup \{yv_iz = yv_iu_izv_iz : \ i \in I\}.$$

We claim that form any involution band S we have $S \in \mathcal{V}^{\mathrm{tw}}$ if and only if S satisfies all the identities from Θ^{tw} (i.e. that $\mathcal{V}^{\mathrm{tw}}$ coincides with the variety defined by Θ^{tw}).

So, let $S \in \mathcal{V}^{\mathrm{tw}}$ and let $\alpha : X \to S$ be an arbitrary valuation of variables, which assigns to each semigroup word w an element $w^\alpha \in S$. Then we have $S/\mathcal{D}' \in \mathcal{V}$, and thus for each $i \in I$, $S/\mathcal{D}'$ satisfies $u_i = v_i$. In other words, we have $(u_i^\alpha, v_i^\alpha) \in \mathcal{D}'$. By Proposition 1, for arbitrary $p, q \in S$ we have $pu_i^\alpha q = pu_i^\alpha pv_i^\alpha u_i^\alpha q$ and $pv_i^\alpha q = pv_i^\alpha u_i^\alpha qv_i^\alpha q$. This means that S satisfies the identities indicated in the definition of Θ^{tw}. However, note that the above argument can be reversed without any trouble at all, so that we obtain the converse implication. Thus, the theorem is proved. ■

Now we give the central result of this note, the aforementioned representation theorem.

Theorem 7 *For any $i \geq 0$, every involution $\mathcal{B}_{i+2}$-band is isomorphic to a twisted spined square of some $\mathcal{LB}_{i+2}$-band over some involution $\mathcal{B}_i$-band. In short, $(\mathcal{B}_i^*)^{\mathrm{tw}} = \mathcal{B}_{i+2}^*$. Also, for all $i \geq 0$ we have $(\mathcal{B}_i^{\mathrm{reg}})^{\mathrm{tw}} = \mathcal{B}_{i+2}^{\mathrm{reg}}$.*

Proof. The first assertion of the theorem follows immediately from Theorem 4 and Lemma 5.

For the second assertion assume that $\mathcal{V} \leq \mathcal{B}^{\mathrm{reg}}$. Since $\mathcal{V}$ satisfies the identity $x = xx^*x$, the proof of the above theorem implies that $\mathcal{V}^{\mathrm{tw}}$ satisfies $yxz = yxy(xx^*x)xz = yxyxx^*xz = yxx^*xz$. By identifying y, z and x we obtain $x = xx^*x$, so that $\mathcal{V}^{\mathrm{tw}} \leq \mathcal{B}^{\mathrm{reg}}$. With these facts, we obtain the required conclusion by applying Theorem 4 and Lemma 5 once again. ■

This result is a generalization of Theorem 2.2 of Scheiblich [21]. In our notation, this theorem asserts that $(\mathcal{SL}^{\mathrm{reg}})^{\mathrm{tw}} = \mathcal{R}e\mathcal{B}^{\mathrm{reg}}$.

The mapping $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{tw}}$ is obviously monotone, and moreover, we have that $\mathcal{U}^{\mathrm{tw}} \wedge \mathcal{V}^{\mathrm{tw}} = (\mathcal{U} \wedge \mathcal{V})^{\mathrm{tw}}$ holds for any $\mathcal{U}$ and $\mathcal{V}$ (indeed, $S \in (\mathcal{U} \wedge \mathcal{V})^{\mathrm{tw}}$ if and only if $S/\mathcal{D}' \in \mathcal{U} \wedge \mathcal{V}$ — in other words, $S/\mathcal{D}' \in \mathcal{U}$ and $S/\mathcal{D}' \in \mathcal{V}$, which means that $S \in \mathcal{U}^{\mathrm{tw}} \wedge \mathcal{V}^{\mathrm{tw}}$). Further, the twisting operator is extensive, i.e. $\mathcal{V} \leq \mathcal{V}^{\mathrm{tw}}$ holds because if $S \in \mathcal{V}$, then, of course, $S/\mathcal{D}' \in \mathcal{V}$. These observations lead to some easy, but interesting consequences.

Proposition 8 *Let $\mathcal{V}$ be an involution band variety of type $i \geq 2$ and let $2 \leq j \leq i+1$. Then*

(i) $\mathcal{V}^{\mathrm{tw}} \wedge \mathcal{B}^*_j = (\mathcal{V} \wedge \mathcal{B}^*_{j-2})^{\mathrm{tw}}$,

(ii) $\mathcal{V} \leq (\mathcal{V} \wedge \mathcal{B}^*_{i-2})^{\mathrm{tw}}$,

(iii) $(\mathcal{V} \wedge \mathcal{B}^*_{i-2})^{\mathrm{tw}} \vee \mathcal{B}^0_{i+2} \leq \mathcal{V}^{\mathrm{tw}}$, *provided* $\mathcal{V} \neq \mathcal{B}^{\mathrm{reg}}_i$,

(iv) $(\mathcal{B}^0_i)^{\mathrm{tw}} \wedge \mathcal{B}^*_j = (\mathcal{B}^0_{j-2})^{\mathrm{tw}}$.

Proof. The relations (i) and (ii) follow from the above remarks, while (iii) is a consequence of the monotonicity of the twisting operator and Lemma 2.1 of [6] (which in this situation yields that $\mathcal{B}^0_{i+2} \leq \mathcal{V}^{\mathrm{tw}}$). Finally, (iv) follows from (i) and the equality $\mathcal{B}^0_i \wedge \mathcal{B}^*_{j-2} = \mathcal{B}^0_{j-2}$, which is implied by Proposition 2.2 of [6]. ∎

The collected information allows us to draw a rough sketch of the lattice of subvarieties of $\mathcal{B}^*$. Namely, by Lemma 2.1 of [6] cited above, every involution band variety $\mathcal{V}$ of type $i \geq 0$ different from $\mathcal{B}^{\mathrm{reg}}_i$ is contained in the interval $\Gamma_i = [\mathcal{B}^0_i, \mathcal{B}^*_i]$. Therefore, it is essential to consider these intervals. The above proposition show that the intervals Γ_j, $j < i$, serve as a kind of the 'skeleton' for the structure of the upper part of Γ_i (namely, for $[(\mathcal{B}^0_{i-2})^{\mathrm{tw}}, \mathcal{B}^*_i]$). It is the twisting operator which realizes this lifting.

We finish the note by pointing out another set of important varieties of involution bands which are obtained by means of twists. But first we need an auxiliary result.

Lemma 9 *For each (involution) band S, S and $S/\mathcal{D}'$ are based on the same structure (involution) semilattice, that is, $S/\mathcal{D}_S \cong (S/\mathcal{D}')/\mathcal{D}_{S/\mathcal{D}'}$.*

Proof. Clearly, we consider the mapping $\phi : S/\mathcal{D}_S \to (S/\mathcal{D}')/\mathcal{D}_{S/\mathcal{D}'}$ defined by $\phi(a\mathcal{D}_S) = (a\mathcal{D}')\mathcal{D}_{S/\mathcal{D}'}$. It is easy to prove that ϕ is a surjective *-homomorphism (since the relations $\mathcal{D}$ and $\mathcal{D}'$ are *-congruneces on any involution band), once we prove that it is well-defined and injective. In other words, we should verify that $(a, b) \in \mathcal{D}_S$ if and only if $(a\mathcal{D}', b\mathcal{D}') \in \mathcal{D}_{S/\mathcal{D}'}$ for all $a, b \in S$.

Obviously, the first of the these conditions is equivalent to $a = aba$ and $b = bab$, while the latter becomes $a\mathcal{D}' = a\mathcal{D}' \cdot b\mathcal{D}' \cdot a\mathcal{D}' = (aba)\mathcal{D}'$ and $b\mathcal{D}' = (bab)\mathcal{D}'$.

Hence, the direct implication is immediate. Conversely, let $(a, aba), (b, bab) \in \mathcal{D}'$. Then, by Proposition 1, $xay = xax(aba)ay = xabay$ for all $x, y \in S$, and so, by letting $x = y = a$, we obtain $a = aba$. Similarly, we have $b = bab$, and the lemma is proved. ■

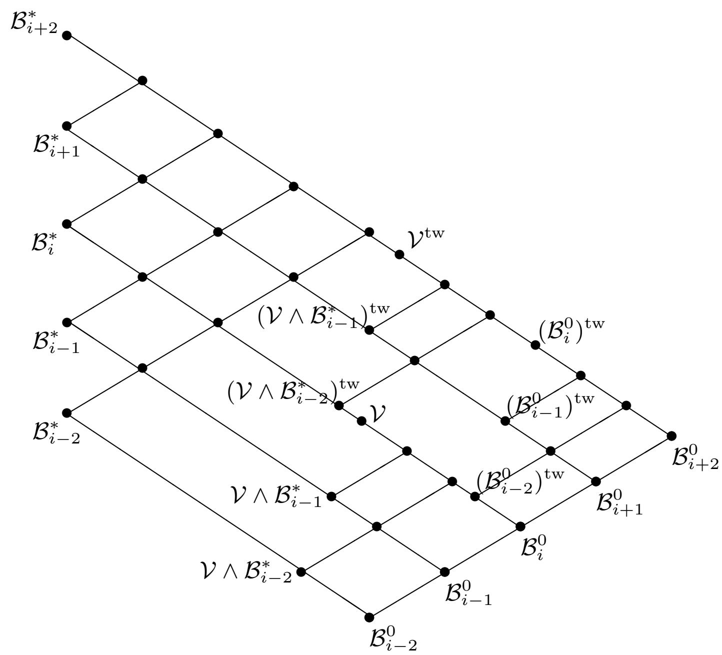

Figure 3. Some varieties of involution bands

Let $\mathcal{V}$ be an involution band variety. Define a sequence of varieties $tw_n(\mathcal{V})$, $n \geq 0$, recursively by $tw_0(\mathcal{V}) = \mathcal{V}$ and $tw_n(\mathcal{V}) = (tw_{n-1}(\mathcal{V}))^{\mathrm{tw}}$ for all $n \geq 1$.

Theorem 10 *Let $\mathcal{X}$ be a nontrivial variety of involution semilattices and let $S \in \mathcal{B}^*_i$ for some $0 \leq i < \infty$. If $i = 2n$, then $S/\mathcal{D} \in \mathcal{X}$ if and only if $S \in tw_n(\mathcal{X})$, while for $i = 2n+1$ we have $S/\mathcal{D} \in \mathcal{X}$ if and only if $S \in tw_n(\mathcal{X} \vee \mathcal{RB}^*)$.*

Consequently, $tw_n(\mathcal{X})$ (resp. $tw_n(\mathcal{X} \vee \mathcal{RB}^*)$) is the largest variety $\mathcal{V}$ of type $2n$ (resp. $2n+1$) such that $\mathcal{V} \wedge \mathcal{SL}^* = \mathcal{X}$.

Proof. We prove the theorem by induction on n. For $n = 0$ and $i = 0$, it suffices to note that for any (involution) semilattice S, $\mathcal{D}_S = \Delta_S$. On the other hand, for $i = 1$, the fact that for a normal involution band S we have $S/\mathcal{D} \in \mathcal{X}$ if and only if $S \in \mathcal{X} \vee \mathcal{RB}^*$ follows from the description of the lattice of all normal involution band varieties, given in [7], and Propositions 4.3, 4.4 and 4.6 of [6] (alternatively, the same conclusion can be extracted from Section 7 of [8]).

Now let $S \in \mathcal{B}_i^*$ for some $i \geq 2$. By the above lemma, $S/\mathcal{D}_S \in \mathcal{X}$ if and only if $(S/\mathcal{D}')/\mathcal{D}_{S/\mathcal{D}'} \in \mathcal{X}$. By the induction hypothesis, the latter condition is equivalent to $S/\mathcal{D}' \in tw_{n-1}(\mathcal{U})$, where $\mathcal{U}$ is $\mathcal{X}$ or $\mathcal{X} \vee \mathcal{RB}^*$ depending on whether i is even or odd, and $n = \lfloor \frac{i}{2} \rfloor$. By definition of the twisting operator, this means that $S \in (tw_{n-1}(\mathcal{U}))^{\mathrm{tw}} = tw_n(\mathcal{U})$, as desired. ∎

By Theorem 3.1 of [6], there are precisely four nontrivial varieties of involution semilattices: $\mathcal{SL}^{\mathrm{reg}}$, $\mathcal{SL}^0$, $\mathcal{SL}' = \mathcal{SL}^{\mathrm{reg}} \vee \mathcal{SL}^0$ (determined by the identity $xx^*y = xx^*y^*$) and $\mathcal{SL}^*$. The mentioned Propositions 4.3, 4.4 and 4.6 of [6] show that if $\mathcal{X}$ is any of these varieties, the condition $S/\mathcal{D} \in \mathcal{X}$ determines a subvariety $\mathcal{B}^*(\mathcal{X})$ of $\mathcal{B}^*$ (and of $\mathcal{B}_i^*$, if we are working only with varieties of type $\leq i$). These propositions provide both structural and equational descriptions of $\mathcal{B}_i^*(\mathcal{X}) = \mathcal{B}^*(\mathcal{X}) \wedge \mathcal{B}_i^*$, $0 \leq i \leq \infty$. Thus, the above theorem adds yet another entry to the list of conditions characterizing these varieties.

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