Remarks on power automata*

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Abstract

The construction of power algebras is an important way for producing new algebras from the given ones. Much information about an algebra can be derived from that concerning its power algebras, and it is interesting to treat the questions such as: What structural properties of A are inherited by its power algebras, and to what extent does the structure of power algebras of A determine that of A? In this paper[1] we treat such questions concerning automata.

1 Introduction and Preliminaries

For any algebra A, every its fundamental operation can be naturally extended to its power set, which gives two new algebras of the same type – the power algebra $\mathcal{P}(A)$ of all subsets of A and the power algebra $\mathcal{P}'(A)$ of nonempty subsets of A. This construction is an important way for producing new algebras from the given ones. Much information about an algebra can be derived from that concerning its power algebras, so it is interesting to treat the questions such as: What structural properties of A are inherited by its power algebras, and to what extent does the structure of power algebras of A determine that of A? One of the most frequently used tools for describing the structure of algebras are identities. It was proved in [15] that identities preserved under formation of power algebras are the linear ones, i.e. identities both of whose sides contain at most one appearance of each variable. Seeing that all automaton identities are linear, it is interesting to make a deeper study of connections between identities satisfied on automata and their power automata, which is the main aim of this paper.

In Section 2 some general properties of power automata are described, and especially, connections between identities satisfied on an automaton and its power automata are considered. The most interesting results are obtained in Section 3.

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By the main result of that section, the direct product of power automata $\mathcal{P}(A_\alpha)$, $\alpha \in Y$, is isomorphic to the power automaton $\mathcal{P}(A)$, where A is a direct sum of automata A_α, $\alpha \in Y$.

The subject of Section 4 are varieties, generalized varieties and pseudovarieties generated by power automata. The study of certain natural operations on classes of rational languages led to investigation of varieties, generalized varieties and pseudovarieties of semigroups and monoids generated by power semigroups and power monoids (see, for example, [10, 16, 14, 11, 8, 2, 3]), and in [16] two power operators $\mathfrak{P}$ and $\mathfrak{P}'$ were introduced, which to any variety (resp. generalized variety, pseudovariety) $\boldsymbol{K}$ assign the varieties (resp. generalized varieties, pseudovarieties) $\mathfrak{P}(\boldsymbol{K})$ and $\mathfrak{P}'(\boldsymbol{K})$ generated by $\{\mathcal{P}(A) \mid A \in \boldsymbol{K}\}$ and $\{\mathcal{P}'(A) \mid A \in \boldsymbol{K}\}$, respectively. It was proved in [8] (see also [3]) that the iteration of the operators $\mathfrak{P}$ and $\mathfrak{P}'$ on pseudovarieties of semigroups and monoids stabilizes in three steps, and in [2] it was shown that on varieties of semigroups $\mathfrak{P}'$ is idempotent and the iteration of $\mathfrak{P}$ stabilizes in two steps. Here we consider similar problems concerning varieties, generalized varieties and pseudovarieties of automata and we prove that in the case of automata the situation is much simpler. Namely, we prove that in every of these three cases, both of these power operators are idempotent, i.e. they are closure operators, and in addition, $\mathfrak{P}'$ is the identity operator, whereas $\mathfrak{P}$ equals the regularization operator.

Automata considered throughout this paper will be automata without outputs in the sense of the definition from the book by F. Gécseg and I. Peák [7]. It is well known that automata without outputs, with the input alphabet X, can be considered as unary algebras of type indexed by X, so the notions such as a *congruence*, *homomorphism*, *generating set* etc., have their usual algebraic meanings (see, for example, [6]). The state set and the input set of an automaton are not necessarily finite. In order to simplify the notations, an automaton with the state set A is also denoted by the same letter A. For any considered automaton A, its input alphabet is denoted by X, and the free monoid over X, the input monoid of A, is denoted by X^*. Under the action of an input word $u \in X^*$, the automaton A goes from a state a into the state denoted by au.

A state a of an automaton A is called a *trap* of A if $au = a$, for every word $u \in X^*$. Automaton all of whose states are traps is called a *discrete automaton* and the class of all discrete automata is denoted by $\mathbf{D}$. By D_2 the discrete automaton with two states is denoted. An automaton A is called a *direct sum* of its subautomata A_α, $\alpha \in Y$, if $A = \bigcup_{\alpha \in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \varnothing$, for every $\alpha, \beta \in Y$ such that $\alpha \neq \beta$. The direct sum of an automaton A and the trivial automaton is called the *trap-extension* of A and denoted by A^t. A word $u \in X^*$ is called a *directing word* of an automaton A if $au = bu$, for every $a, b \in A$, and the set of all directing words of A is denoted by $DW(A)$. If $DW(A)$, then there exists the unique state $d_u \in A$ such that $au = d_u$, for every $a \in A$, and it is called the *u-neck* of A. A state $d \in A$ is called a *neck* of A if it is a u-neck of A, for some $u \in DW(A)$.

If $\boldsymbol{K}$ is a class of automata, then by $\mathbf{H}(\boldsymbol{K})$ (resp. $\mathbf{I}(\boldsymbol{K})$, $\mathbf{S}(\boldsymbol{K})$, $\mathbf{P}(\boldsymbol{K})$, $\mathbf{P_f}(\boldsymbol{K})$, $\mathbf{Pow}(\boldsymbol{K})$) we denote the class of all homomorphic images (resp. isomorphic copies, subautomata, direct products, finite direct products, direct powers) of automata from $\boldsymbol{K}$. A class of closed under the operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ is called a *variety*, a class closed under the operators $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P_f}$ and $\mathbf{Pow}$ is a *generalized variety*, and a class of finite automata closed under the operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P_f}$ is called a *pseudovariety*. For an arbitrary class $\boldsymbol{K}$ of automata, $\mathbf{V}(\boldsymbol{K}) = \mathbf{HSP}(\boldsymbol{K})$ (resp. $\mathbf{G}(\boldsymbol{K}) = \mathbf{HSP_f Pow}(\boldsymbol{K})$) is the least variety (resp. generalized variety) containing $\boldsymbol{K}$ and it is called the *variety* (resp. *generalized variety*) *generated by* $\boldsymbol{K}$, and if $\boldsymbol{K}$ is a class of finite automata, then $\mathbf{Pv}(\boldsymbol{K}) = \mathbf{HSP_f}(\boldsymbol{K})$ is the least pseudovariety containing $\boldsymbol{K}$, and it is called the *pseudovariety generated by* $\boldsymbol{K}$.

A *term* of type X^* is any expression of the form gu, where $u \in X^*$ and $g \notin X^*$ is a symbol which is called a *variable*. A formal equality $s = t$ of two terms s and t is called an *identity*. An identity of the form $gu = gv$ is called *regular*, and an identity of the form $gu = hv$, where g and h are distinct variables, is called *irregular*. An automaton A *satisfies an identity* $s = t$ if s and t give rise to the same state of A for every possible substitution of states of A for the variables appearing in s and t, and A *satisfies a set of identities* Σ, in notation $A \models \Sigma$, if it satisfies every identity from this set. It is well known that a class of automata $\boldsymbol{K}$ is a variety if and only if it is the class of all automata satisfying some set of identities Σ, and then $\boldsymbol{K}$ is said to be *defined by the set of identities* Σ. A variety $\boldsymbol{K}$ is called *regular* if it can be defined by a set of regular identities. Otherwise it is called an *irregular variety*. If $\boldsymbol{K}$ is a variety defined by a set of identities Σ, then the variety $\mathbf{R}(\boldsymbol{K})$ defined by the set Σ_R of all regular identities from Σ is the least regular variety containing $\boldsymbol{K}$, and it is called the *regularization* of the variety $\boldsymbol{K}$.

A family $\{H_i\}_{i\in I}$ of sets is said to be *directed* if for every $i, j \in I$ there exists $k \in I$ such that $H_i \subseteq H_k$ and $H_j \subseteq H_k$. It is known (see [1]) that a class of automata is a generalized variety if and only if it can be represented as the union of a directed family of varieties. If $\boldsymbol{K}$ can be represented as the union of a directed family of regular varieties, then it is said to be a *regular generalized variety*. Otherwise it is called *irregular*. It was proved in [4] that a generalized variety of automata $\boldsymbol{K}$ is regular if and only if $D_2 \in \boldsymbol{K}$. For every generalized variety $\boldsymbol{K}$ there exists the least regular generalized variety $\mathbf{R}(\boldsymbol{K})$ containing $\boldsymbol{K}$, called its *regularization*. It was proved in [1] that a class of finite automata is a pseudovariety if and only if it is the class the class of all finite automata from some generalized variety of automata. If $\boldsymbol{K}$ is the class of all finite automata from some regular generalized variety, then it is said to be a *regular pseudovariety*. Otherwise it is called *irregular*. For a pseudovariety $\boldsymbol{K}$, by $\mathbf{R}(\boldsymbol{K})$ we denote the *regularization* of $\boldsymbol{K}$, i.e. the least regular pseudovariety containing $\boldsymbol{K}$.

For undefined notions and notations we refer to the books [7] and [6].

2 Some General Properties

For an automaton A, by $\mathcal{P}(A)$ we denote the set of all subsets of A, i.e. its *power set*, and by $\mathcal{P}'(A)$ the set of all nonempty subsets of A. If for $H \in \mathcal{P}'(A)$ and

$u \in X^*$ we set $Hu = \{au \,|\, a \in H\}$, and $\varnothing u = \varnothing$, then this defines transitions on $\mathcal{P}(A)$ and $\mathcal{P}'(A)$ so that $\mathcal{P}(A)$ and $\mathcal{P}'(A)$ become automata called the *power automata* of A. It can be seen easily that $\mathcal{P}(A)$ is a trap-extension of and $\mathcal{P}'(A)$. For a class $\boldsymbol{K}$ of automata we set

$$\mathcal{P}(\boldsymbol{K}) = \{\mathcal{P}(A) \,|\, A \in \boldsymbol{K}\} \quad \text{and} \quad \mathcal{P}'(\boldsymbol{K}) = \{\mathcal{P}'(A) \,|\, A \in \boldsymbol{K}\}.$$

In this section we describe some general properties of power automata.

Lemma 2.1 *Every automaton A can be embedded into its power automata $\mathcal{P}'(A)$ and $\mathcal{P}(A)$.*

Proof. The mapping $a \mapsto \{a\}$ is an embedding of A into $\mathcal{P}'(A)$. It is also an embedding into $\mathcal{P}(A)$, since $\mathcal{P}(A)$ is a trap-extension of $\mathcal{P}'(A)$. ∎

Lemma 2.2 *Let B be a subautomaton of an automaton A. Then $\mathcal{P}'(B)$ is a subautomaton of $\mathcal{P}'(A)$ and $\mathcal{P}(B)$ is a subautomaton of $\mathcal{P}(A)$.*

In view of Lemma 2.1, in the next theorem the automaton A is treated as a subautomaton of its power automata $\mathcal{P}'(A)$ and $\mathcal{P}(A)$.

Theorem 2.1 *Every mapping φ of an automaton A into an automaton B can be extended to a mapping $\widehat{\varphi}$ of $\mathcal{P}(A)$ into $\mathcal{P}(B)$ such that the following conditions hold:*

(a) *$\widehat{\varphi}$ maps $\mathcal{P}'(A)$ into $\mathcal{P}'(B)$.*
(b) *φ is a homomorphism, one-to-one or onto if and only if $\widehat{\varphi}$ has the same property.*

Proof. For $H \in \mathcal{P}'(A)$ let $H\widehat{\varphi} = \{a\varphi \,|\, a \in H\}$ and let $\varnothing\widehat{\varphi} = \varnothing$. It is evident that $\widehat{\varphi}$ is an extension of φ and that (a) holds.

(b) Let φ be a homomorphism and let $H \in \mathcal{P}'(A)$. Then

$$\begin{aligned} b \in (H\widehat{\varphi})u \ &\Leftrightarrow\ b = (a\varphi)u, \text{ for some } a \in H \\ &\Leftrightarrow\ b = (au)\varphi), \text{ for some } a \in H \\ &\Leftrightarrow\ b \in (Hu)\widehat{\varphi}). \end{aligned}$$

Thus $\widehat{\varphi}$ is a homomorphism.

Let φ be one-to-one. Suppose that $H_1\widehat{\varphi} = H_2\widehat{\varphi}$ for some $H_1, H_2 \in \mathcal{P}(A)$. If $H_1\widehat{\varphi} = H_2\widehat{\varphi} = \varnothing$, then $H_1 = H_2 = \varnothing$. Let $H_1\widehat{\varphi} = H_2\widehat{\varphi} \neq \varnothing$. Then $H_1, H_2 \in \mathcal{P}'(A)$. For any $a \in H_1$ we have that $a\varphi \in H_1\widehat{\varphi} = H_2\widehat{\varphi}$, which means that $a\varphi = b\varphi$, for some $b \in H_2$. Since φ is one-to-one, then $a = b \in H_2$, so we have proved that $H_1 \subseteq H_2$. Similarly we prove that $H_2 \subseteq H_1$. Hence $H_1 = H_2$ and we have proved that $\widehat{\varphi}$ is also one-to-one.

Finally, suppose that φ maps A onto B. Let $H' \in \mathcal{P}'(B)$ and let $H = \{a \in A \,|\, a\varphi \in H'\}$. Since φ is onto, then $H' = \{a\varphi \,|\, a \in H\} = H\widehat{\varphi}$. Therefore, $\widehat{\varphi}$ maps $\mathcal{P}(A)$ onto $\mathcal{P}(B)$.

Conversely, if $\widehat{\varphi}$ is a homomorphism or one-to-one, then φ has the same property, as a restriction of $\widehat{\varphi}$. If $\widehat{\varphi}$ maps $\mathcal{P}(A)$ onto $\mathcal{P}(B)$, then for any $b \in B$ there exists $H \in \mathcal{P}'(A)$ such that $H\widehat{\varphi} = \{b\}$, that is $a\varphi = b$, for each $a \in H$, so we have that φ maps A onto B. ∎

Next we consider identities satisfied on power automata.

Lemma 2.3 *Let A be any automaton. An automaton identity $s = t$ is satisfied on A^t if and only if it is regular and it is satisfied on A.*

Lemma 2.4 ([4], [9]) *An irregular identity $gu = hv$ is satisfied on an automaton A if and only if $u, v \in DW(A)$ and they determine the same neck of A.*

By the previous two lemmas we obtain the following:

Theorem 2.2 *Let A be an arbitrary automaton and let $s = t$ be an arbitrary automaton identity. Then*

(a) *The identity $s = t$ is satisfied on the power automaton $\mathcal{P}'(A)$ if and only if it is satisfied on A.*

(b) *The identity $s = t$ is satisfied on the power automaton $\mathcal{P}(A)$ if and only if it is regular and it is satisfied on A.*

Proof. (a) Let A satisfies $s = t$. If $s = t$ is a regular identity, i.e. if it has the form $gu = gv$, for some $u, v \in X^*$, then $au = av$ for each $a \in A$, and for every $H \in \mathcal{P}'(A)$ we have that

$$Hu = \{au \,|\, a \in H\} = \{av \,|\, a \in H\} = Hv,$$

which means that $\mathcal{P}'(A)$ satisfies $gu = gv$. On the other hand, if $s = t$ is an irregular identity, i.e. it has the form $gu = hv$, where $g \neq v$ and $u, v \in X^*$, then $u, v \in DW(A)$ and $d_u = d_v$, by Lemma 2.4, and for arbitrary $H_1, H_2 \in \mathcal{P}'(A)$ we have that

$$H_1u = \{d_u\} = \{d_v\} = H_2v,$$

which means that $\mathcal{P}'(A)$ satisfies $gu = hv$. This completes the proof for (a).

(b) This follows by Lemma 2.3 and the fact that $\mathcal{P}(A)$ is the trap-extension of $\mathcal{P}'(A)$. ∎

The *transition semigroup* of an automaton A is defined as the subsemigroup of the full transformation semigroup of A generated by all mappings $\eta_u : a \mapsto au$, $u \in X^*$. It is known that this semigroup is isomorphic to the factor semigroup of the free semigroup X^* with respect to the *Myhill congruence* μ_A defined on X^* by: $(u, v) \in \mu_A \;\Leftrightarrow\; au = av$, for every $a \in A$. This equivalent way for defining the transition semigroup is used in the proof of the following consequence of Theorem 2.2.

Now we are ready to state the following consequence of Theorem 2.2:

Corolarry 2.1 *Automata A, $\mathcal{P}'(A)$ and $\mathcal{P}(A)$ have isomorphic transition semigroups, for every automaton A.*

Proof. It is clear that the Myhill congruence of an automaton consists of all pairs of words (u, v) such that the regular identity $gu = gv$ is satisfied on this automaton. By this fact and by Theorem 2.2 it follows that the automata A, $\mathcal{P}'(A)$ and $\mathcal{P}(A)$ have the same Myhill congruence, so they have isomorphic transition semigroups. ∎

3 Direct Products of Power Automata

In this section we describe some properties of direct products of power automata.

The first theorem of the section establishes a very interesting connection between direct products of power automata and direct sums of automata.

Theorem 3.1 *Let A_α, $\alpha \in Y$, be any family of automata. An automaton B is isomorphic to the direct product of power automata $\mathcal{P}(A_\alpha)$, $\alpha \in Y$, if and only if it is isomorphic to the power automaton $\mathcal{P}(A)$, where A is the direct sum of automata A_α, $\alpha \in Y$.*

Proof. Let A be the direct sum of automata A_α, $\alpha \in Y$. We have to prove that the power automaton $\mathcal{P}(A)$ is isomorphic to the direct product of automata $\mathcal{P}(A_\alpha)$, $\alpha \in Y$. Define a mapping

$$\varphi : \mathcal{P}(A) \to \prod_{\alpha\in Y} \mathcal{P}(A_\alpha)$$

as follows. For $H \in \mathcal{P}(A)$ let

$$H\varphi = (H_\alpha)_{\alpha\in Y}, \qquad \text{where } H_\alpha = H \cap A_\alpha,\ \alpha \in Y.$$

By a straightforward verification we obtain that φ is a one-to-one and onto. It remains to check that φ is a homomorphism. Indeed, let $H \in \mathcal{P}(A)$ and $u \in X^*$, let $Hu = H'$, and for any $\alpha \in Y$ let $H_\alpha = H \cap A_\alpha$ and $H'_\alpha = H' \cap A_\alpha$. Then for any $\alpha \in Y$ we have that

$$\begin{aligned} b \in H_\alpha u \ &\Leftrightarrow\ b = au, \text{ for some } a \in H_\alpha \\ &\Leftrightarrow\ b = au, \text{ for some } a \in H \cap A_\alpha \\ &\Leftrightarrow\ b \in Hu \text{ and } \ b \in A_\alpha \\ &\Leftrightarrow\ b \in H' \text{ and } \ b \in A_\alpha \\ &\Leftrightarrow\ b \in H'_\alpha. \end{aligned}$$

Thus

$$(H\varphi)u = (H_\alpha u)_{\alpha\in Y} = (H'_\alpha)_{\alpha\in Y} = H'\varphi = (Hu)\varphi.$$

This completes the proof of the theorem. ∎

Corolarry 3.1 *Let $\boldsymbol{K}$ be a class of automata which is closed under isomorphic copies and (finite) direct sums. Then the class $\mathcal{P}(\boldsymbol{K})$ is closed under (finite) direct products.*

Proof. Let an automaton P be isomorphic to the direct product of automata P_α, $\alpha \in Y$, where $P_\alpha \in \mathcal{P}(\boldsymbol{K})$, for each $\alpha \in Y$. This means that $P_\alpha \cong \mathcal{P}(A_\alpha)$ for some $A_\alpha \in \boldsymbol{K}$. Without loss of generality we can assume that $A_\alpha \cap A_\beta = \varnothing$ for $\alpha \neq \beta$. Let A be the direct sum of automata A_α, $\alpha \in Y$. Then by Theorem 3.1 we have that the power automaton $\mathcal{P}(A)$ is isomorphic to the direct product of automata P_α, $\alpha \in Y$, that is to the automaton P. Thus $P \in \mathcal{P}(\boldsymbol{K})$, which was to be proved. ∎

Corolarry 3.2 *Let $\boldsymbol{K}$ be a class of automata which is closed under finite direct products and contains the class* $\mathbf{D}$ *of discrete automata. Then the class $\mathcal{P}(\boldsymbol{K})$ is closed under direct powers.*

Proof. Let an automaton A be a direct power of some automaton $\mathcal{P}(B)$, where $B \in \boldsymbol{K}$, i.e. $A \cong (\mathcal{P}(B))^Y$, for some nonempty set Y. Define the transitions on Y by: $\alpha u = \alpha$, for every $\alpha \in Y$ and $u \in X^*$. Then Y is a discrete automaton, and if we set $B_\alpha = B \times \{\alpha\}$, then B_α is an automaton isomorphic to B, for each $\alpha \in Y$. Let C be the direct sum of automata B_α, $\alpha \in Y$. It is evident that $C \cong B \times Y$, and since $B, Y \in \boldsymbol{K}$ and $\boldsymbol{K}$ is closed under finite direct products, by the assumptions of the corollary, then $C \in \boldsymbol{K}$. On the other hand, by Theorem 3.1 it follows that

$$\mathcal{P}(C) \cong \prod_{\alpha \in Y} \mathcal{P}(B_\alpha) \cong (\mathcal{P}(B))^Y \cong A.$$

Thus $A \in \mathcal{P}(\boldsymbol{K})$, so we have proved that the class $\mathcal{P}(\boldsymbol{K})$ is closed under direct powers. ∎

Theorem 3.2 *Let A_α, $\alpha \in Y$, be any family of automata. Then the direct product of automata $\mathcal{P}'(A)$, $\alpha \in Y$, is isomorphic to a subautomaton of $\mathcal{P}'(A)$, where A is the direct product of automata A_α, $\alpha \in Y$.*

Proof. It can be easily verified that the mapping $(H_\alpha)_{\alpha \in Y} \mapsto \prod_{\alpha \in Y} H_\alpha$ is an embedding of the direct product of automata $\mathcal{P}'(A_\alpha)$, $\alpha \in Y$, into the automaton $\mathcal{P}'(A)$. ∎

4 Varieties, Generalized Varieties and Pseudovarieties Generated by Power Automata

In this section we study closure properties of varieties, generalized varieties and pseudovarieties generated by power automata.

First we prove the following lemma.

Lemma 4.1 *Let $\boldsymbol{K}$ be an arbitrary class of automata. Then*

$$\boldsymbol{K} \subseteq \mathbf{S}(\mathcal{P}'(\boldsymbol{K})) \subseteq \mathbf{S}(\mathcal{P}(\boldsymbol{K})).$$

Proof. Consider an arbitrary $A \in \boldsymbol{K}$. Then $A \cong A'$, where A' is a subautomaton of $\mathcal{P}'(A)$, so

$$A \in \mathbf{IS}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{SI}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}'(\boldsymbol{K})).$$

Thus $\boldsymbol{K} \subseteq \mathbf{S}(\mathcal{P}'(\boldsymbol{K}))$, and clearly $\mathbf{S}(\mathcal{P}'(\boldsymbol{K})) \subseteq \mathbf{S}(\mathcal{P}(\boldsymbol{K}))$. ∎

In the next theorem we consider varieties generated by power automata.

Theorem 4.1 *Let $\boldsymbol{K}$ be an arbitrary variety of automata. Then*

(a) $\mathbf{V}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}'(\boldsymbol{K})) = \boldsymbol{K}$.
(b) $\mathbf{V}(\mathcal{P}(\boldsymbol{K})) = \mathbf{SP}(\mathcal{P}(\boldsymbol{K})) = \mathbf{R}(\boldsymbol{K})$.

If $\boldsymbol{K}$ is a regular variety, then

(c) $\mathbf{V}(\mathcal{P}(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}(\boldsymbol{K})) = \boldsymbol{K}$.

Proof. (a) Let $A \in \mathcal{P}'(\boldsymbol{K})$, i.e. let $A \cong \mathcal{P}'(B)$, for some $B \in \boldsymbol{K}$. Moreover, let the variety $\boldsymbol{K}$ be defined by a set of identities Σ. Then $B \models \Sigma$, whence $A \models \Sigma$, by Theorem 2.2, so we have that $A \in \boldsymbol{K}$. Thus $\mathcal{P}'(\boldsymbol{K}) \subseteq \boldsymbol{K}$ and by this it follows

$$\mathbf{V}(\mathcal{P}'(\boldsymbol{K})) \subseteq \boldsymbol{K}. \tag{1}$$

On the other hand, by Lemma 4.1 we have that

$$\boldsymbol{K} \subseteq \mathbf{S}(\mathcal{P}'(\boldsymbol{K})) \subseteq \mathbf{HSP}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{V}(\mathcal{P}'(\boldsymbol{K})). \tag{2}$$

Therefore, by (1) and (2) it follows that (a) holds.

(b) It is evident that

$$\mathbf{SP}(\mathcal{P}(\boldsymbol{K})) \subseteq \mathbf{HSP}(\mathcal{P}(\boldsymbol{K})) = \mathbf{V}(\mathcal{P}(\boldsymbol{K})). \tag{3}$$

To prove that

$$\mathbf{V}(\mathcal{P}(\boldsymbol{K})) \subseteq \mathbf{R}(\boldsymbol{K}) \tag{4}$$

consider an arbitrary $A \in \mathcal{P}(\boldsymbol{K})$. Then $A \cong \mathcal{P}(B)$ for some $B \in \boldsymbol{K}$. Let Σ be the set of all identities satisfied in the variety $\boldsymbol{K}$ and let Σ_R be the set of all regular identities from Σ. By $B \in \boldsymbol{K}$ it follows that $B \models \Sigma$, and by Theorem 2.2 we have that $\mathcal{P}(B) \models \Sigma_R$, i.e. $A \models \Sigma_R$. Therefore, $A \in [\Sigma_R] = \mathbf{R}(\boldsymbol{K})$, which yields $\mathcal{P}(\boldsymbol{K}) \subseteq \mathbf{R}(\boldsymbol{K})$, and hence (4) holds.

It remains to prove

$$\mathbf{R}(\boldsymbol{K}) \subseteq \mathbf{SP}(\mathcal{P}(\boldsymbol{K})). \tag{5}$$

To prove this inclusion consider an arbitrary $A \in \mathbf{R}(\boldsymbol{K})$. It was proved in [12] (see also [4]) that $\mathbf{R}(\boldsymbol{K})$ consists of automata which are direct sums of automata from $\boldsymbol{K}$, so A is a direct sum of automata A_α, $\alpha \in Y$, such that $A_\alpha \in \boldsymbol{K}$, for each $\alpha \in Y$. By Theorem 3.1 it follows that $\mathcal{P}(A) \cong \prod_{\alpha\in Y} \mathcal{P}(A_\alpha)$, so $\mathcal{P}(A) \in \mathbf{IP}(\mathcal{P}(\boldsymbol{K}))$. On the other hand, $A \cong A'$, where A' is a subautomaton of $\mathcal{P}(A)$ so that

$$A \in \mathbf{ISIP}(\mathcal{P}(\boldsymbol{K})) = \mathbf{SPI}(\mathcal{P}(\boldsymbol{K})) = \mathbf{SP}(\mathcal{P}(\boldsymbol{K}))$$

By this it follows that (5) holds. Now, the equations (3), (4) and (5) yield the assertion (b).

The assertion (c) follows by (b) and Lemma 4.1. ■

Next we consider generalized varieties generated by power automata.

Theorem 4.2 *Let $\boldsymbol{K}$ be an arbitrary generalized variety of automata. Then*

(a) $\mathbf{G}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}'(\boldsymbol{K})) = \boldsymbol{K}$.
(b) $\mathbf{G}(\mathcal{P}(\boldsymbol{K})) = \mathbf{R}(\boldsymbol{K})$.

If $\boldsymbol{K}$ is a regular generalized variety, then

(c) $\mathbf{G}(\mathcal{P}(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}(\boldsymbol{K})) = \boldsymbol{K}$.

Proof. (a) Let $\boldsymbol{K}$ be represented as the union of a directed family of varieties $\{\boldsymbol{V}_i\}_{i\in I}$. By Theorem 4.1 it follows that

$$\mathcal{P}'(\boldsymbol{K}) = \bigcup_{i\in I} \mathcal{P}'(\boldsymbol{V}_i) \subseteq \bigcup_{i\in I} \boldsymbol{V}_i = \boldsymbol{K},$$

whence

$$\mathbf{G}(\mathcal{P}'(\boldsymbol{K})) \subseteq \boldsymbol{K}. \tag{6}$$

On the other hand, by Lemma 4.1 we have that

$$\boldsymbol{K} \subseteq \mathbf{S}(\mathcal{P}'(\boldsymbol{K})) \subseteq \mathbf{HSP_f Pow}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{G}(\mathcal{P}'(\boldsymbol{K})),$$

which with (6) gives the assertion (a).

(b) We have that $D_2 \cong \mathcal{P}(A)$, where A is an automaton with only one state, so $D_2 \in \mathcal{P}(\mathbf{O}) \subseteq \mathcal{P}(\boldsymbol{K}) \subseteq \mathbf{G}(\mathcal{P}(\boldsymbol{K}))$. According to a result proved in [4], a generalized variety is regular if and only if it contains D_2, so we conclude that $\mathbf{G}(\mathcal{P}(\boldsymbol{K}))$ is a regular generalized variety, and hence $\mathbf{R}(\boldsymbol{K}) \subseteq \mathbf{G}(\mathcal{P}(\boldsymbol{K}))$.

It remains to prove the opposite inclusion. Let $\mathbf{R}(\boldsymbol{K})$ be the union of a directed family of regular varieties $\{\boldsymbol{V}_i\}_{i\in I}$, and let $A \in \mathcal{P}(\boldsymbol{K})$, i.e. $A \cong \mathcal{P}(B)$, for some $B \in \boldsymbol{K}$. Then $B \in \mathbf{R}(\boldsymbol{K})$, whence $B \in \boldsymbol{V}_i$, for some $i \in I$, so

$$A \in \mathcal{P}(\boldsymbol{V}_i) \subseteq \mathbf{V}(\mathcal{P}(\boldsymbol{V}_i)) = \boldsymbol{V}_i \subseteq \mathbf{R}(\boldsymbol{K}),$$

by Theorem 4.1, because $\boldsymbol{V}_i$ is a regular variety. Therefore, we have obtained that $\mathcal{P}(\boldsymbol{K}) \subseteq \mathbf{R}(\boldsymbol{K})$, whence $\mathbf{G}(\mathcal{P}(\boldsymbol{K})) \subseteq \mathbf{R}(\boldsymbol{K})$, which was to be proved.

The claim (c) follows by (b) and Lemma 4.1. ∎

Finally, we study pseudovarieties generated by power automata.

Theorem 4.3 *Let $\boldsymbol{K}$ be an arbitrary pseudovariety of automata. Then*

(a) $\mathbf{P}(\mathcal{P}'(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}'(\boldsymbol{K})) = \boldsymbol{K}$.
(b) $\mathbf{P}(\mathcal{P}(\boldsymbol{K})) = \mathbf{SP_f}(\mathcal{P}(\boldsymbol{K})) = \mathbf{R}(\boldsymbol{K})$.

If $\boldsymbol{K}$ is a regular pseudovariety, then

(c) $\mathbf{P}(\mathcal{P}(\boldsymbol{K})) = \mathbf{S}(\mathcal{P}(\boldsymbol{K})) = \boldsymbol{K}$.

Proof. The proof of this theorem is similar to the proof of Theorem 4.1. ∎

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