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SOME INEQUALITIES FOR FUNCTIONS OF BOUNDED VARIATION WITH APPLICATIONS TO LANDAU TYPE RESULTS

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Abstract. Some inequalities for functions of bounded variation that provide reverses for the inequality between the integral mean and the $p-$norm for $p \in [1, \infty]$ are established. Applications related to the celebrated Landau inequality between the norms of the derivatives of a function are also pointed out.

1. Introduction

The following inequality holding on finite intervals is well known.

If $f : [a, b] \to \mathbb{R}$ is essentially bounded, then f is integrable on $[a, b]$ and

$$\frac{1}{b-a}\left|\int_a^b f(t)\,dt\right| \le \|f\|_{[a,b],\infty} \tag{1.1}$$

where $\|f\|_{[a,b],\infty} := ess\sup_{t\in[a,b]} |f(t)|$.

The corresponding version in terms of $p-$norms, is the following Hölder type inequality

$$\frac{1}{(b-a)^{1-\frac{1}{p}}}\left|\int_a^b f(t)\,dt\right| \le \|f\|_{[a,b],p}, \qquad p \ge 1, \tag{1.2}$$

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provided $f \in L_p[a,b]$, where

$$\|f\|_{[a,b],p} := \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}}, \qquad p \geq 1.$$

In the first part of this paper we point out some reverse inequalities for (1.1) and (1.2) in the case of functions of bounded variation. These results are then employed in obtaining some Landau type inequalities.

For the latter, recall that if $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and if $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, $p \in [1,\infty]$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of the function f, such that

$$\left\|f'\right\|_{p,I} \leq C_p(I)\, \|f\|_{p,I}^{\frac{1}{2}} \left\|f''\right\|_{p,I}^{\frac{1}{2}}, \tag{1.3}$$

where $\|\cdot\|_{p,I}$ is the $p-$norm on the interval I.

The investigation of such inequalities was initiated by E. Landau [8] in 1914. He considered the case $p = \infty$ and proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2}, \tag{1.4}$$

are the best constant for which (1.3) holds.

For some classical and recent results related to Landau inequality, see [1],[4] and [5]-[11].

2. Some Reverse Inequalities on Bounded Intervals

The following result for functions of bounded variation holds.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then*

$$\|f\|_{[a,b],\infty} \leq \frac{1}{b-a}\left|\int_a^b f(t)\, dt\right| + \bigvee_a^b (f). \tag{2.1}$$

The multiplicative constant 1 in front of $\bigvee_a^b (f)$ cannot be replaced by a smaller quantity.

Proof. We apply the following Ostrowski type inequality obtained by the author in [2] (see also [3]):

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \leq \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right] \bigvee_a^b (f) \tag{2.2}$$

for any $x \in [a,b]$. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Taking the supremum in (2.2) over $x \in [a,b]$, we get

$$\begin{aligned} \left\|f - \frac{1}{b-a}\int_a^b f(t)\, dt\right\|_{[a,b],\infty} &\leq \sup_{x\in[a,b]} \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right] \bigvee_a^b (f) \\ &= \bigvee_a^b (f). \end{aligned} \tag{2.3}$$

Now, by the triangle inequality applied for the sup-norm $\|\cdot\|_{\infty}$, we get

$$\begin{aligned}\|f\|_{[a,b],\infty} &\leq \left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],\infty}+\left|\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left|\int_a^b f(t)\,dt\right|+\bigvee_a^b(f)\end{aligned}$$

and the inequality (2.1) is proved.

To prove the sharpness of the constant 1, assume that the following inequality holds

$$\|f\|_{[a,b],\infty} \leq \frac{1}{b-a}\left|\int_a^b f(t)\,dt\right|+C\bigvee_a^b(f) \tag{2.4}$$

with a $C>0$.

Consider the function $f_0:[a,b]\to\mathbb{R}$,

$$f_0(t)=\begin{cases}0, & t\in[a,b)\\ 1, & t=b.\end{cases}$$

Then f_0 is of bounded variation on $[a,b]$ and

$$\|f_0\|_{[a,b],\infty}=1, \qquad \int_a^b f_0(t)\,dt=0 \quad \text{and} \quad \bigvee_a^b(f_0)=1.$$

For this choice, (2.4) becomes $C\geq 1$, proving the sharpness of the constant. □

The corresponding result for $p-$norms, where $p\geq 1$, is embodied in the following theorem.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then for $p\geq 1$ one has the inequality*

$$\|f\|_{[a,b],p} \leq \frac{1}{(b-a)^{1-\frac{1}{p}}}\left|\int_a^b f(t)\,dt\right|+\frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{p}}\left(2^{p+1}-1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\bigvee_a^b(f). \tag{2.5}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. Taking the $p-$norm in (2.2), we deduce

$$\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}\leq\bigvee_a^b(f)\,I_p,$$

where

$$I_p:=\left(\int_a^b\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]^p dx\right)^{\frac{1}{p}}, \quad p\geq 1.$$

We observe that

$$\begin{aligned} I_p &:= \left(\int_a^{\frac{a+b}{2}} \left[\frac{1}{2} + \frac{\frac{a+b}{2} - x}{b-a} \right]^p dx + \int_{\frac{a+b}{2}}^b \left[\frac{1}{2} + \frac{x - \frac{a+b}{2}}{b-a} \right]^p dx \right)^{\frac{1}{p}} \\ &= \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} (b-x)^p \, dx + \int_{\frac{a+b}{2}}^b (x-a)^p \, dx \right] \\ &= \frac{(b-a)^{\frac{1}{p}} \left(2^{p+1} - 1\right)^{\frac{1}{p}}}{2 (p+1)^{\frac{1}{p}}}, \quad p \geq 1. \end{aligned}$$

Using the triangle inequality for the $p-$norm $\|\cdot\|_p$, we get

$$\begin{aligned} \|f\|_{[a,b],p} &\leq \left\| f - \frac{1}{b-a} \int_a^b f(t)\, dt \right\|_{[a,b],p} + \left\| \frac{1}{b-a} \int_a^b f(t)\, dt \right\|_{[a,b],p} \\ &\leq \frac{(b-a)^{\frac{1}{p}} \left(2^{p+1} - 1\right)^{\frac{1}{p}}}{2 (p+1)^{\frac{1}{p}}} \bigvee_a^b (f) + (b-a)^{\frac{1}{p}} \left| \frac{1}{b-a} \int_a^b f(t)\, dt \right| \end{aligned}$$

and the inequality (2.5) is obtained.

Now, assume that (2.5) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$\|f\|_{[a,b],p} \leq \frac{1}{(b-a)^{1-\frac{1}{p}}} \left| \int_a^b f(t)\, dt \right| + D \cdot \frac{(b-a)^{\frac{1}{p}} \left(2^{p+1} - 1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \bigvee_a^b (f). \tag{2.6}$$

Consider the function $f_0 : [a,b] \to \mathbb{R}$ with $a = 0$ and $b > 1$ given by

$$f_0(t) = \begin{cases} 0, & \text{if } t \in [0, b-1] \\ 1, & \text{if } t \in (b-1, b]. \end{cases}$$

This function is of bounded variation on $[a,b]$ and

$$\|f\|_{[a,b],p} = 1, \qquad \int_a^b f(t)\, dt = 1 \quad \text{and} \quad \bigvee_a^b (f) = 1$$

and then, by (2.6), we deduce

$$1 \leq \frac{1}{b^{1-\frac{1}{p}}} + D \frac{b^{\frac{1}{p}} \left(2^{p+1} - 1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}, \quad b > 1,\ p \geq 1$$

giving

$$b^{1-\frac{1}{p}} \leq 1 + D \cdot b \frac{\left(2^{p+1} - 1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}. \tag{2.7}$$

Denote

$$q := \frac{\left(2^{p+1} - 1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}.$$

Then

$$\ln q = \frac{\ln\left(2^{p+1}-1\right) - \ln(p+1)}{p}.$$

We observe, by L'Hospital theorem that

$$\lim_{p\to\infty}\left[\frac{\ln\left(2^{p+1}-1\right)}{p}\right] = \lim_{p\to\infty}\frac{\left[\ln\left(2^{p+1}-1\right)\right]'}{(p)'} = \lim_{p\to\infty}\frac{\left(2^{p+1}-1\right)'}{2^{p+1}-1} = \ln 2$$

and

$$\lim_{p\to\infty}\left[\frac{\ln(p+1)}{p}\right] = 0,$$

consequently

$$\lim_{p\to\infty} q = 2.$$

Taking the limit over $p\to\infty$ in (2.7), we deduce

$$b \le 1 + 2Db, \quad \text{for} \quad b > 1$$

from where we get

$$D \ge \frac{b-1}{2b}, \quad b > 1. \tag{2.8}$$

Taking the limit over $b\to\infty$ in (2.8) we conclude that $D \ge \frac{1}{2}$, showing that the constant $\frac{1}{2}$ in (2.5) cannot be replaced by a smaller quantity in (2.5). □

3. Some Inequalities of Landau Type on Unbounded Intervals

The following technical lemma will be used in the following (see also [4]).

Lemma 1. *Let $C, D > 0$ and $r, u \in (0,1]$. Consider the function $g_{r,u} : (0,\infty) \to (0,\infty)$ given by*

$$g_{r,u}(\lambda) = \frac{C}{\lambda^u} + D\lambda^r. \tag{3.1}$$

Define

$$\lambda_0 := \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0,\infty). \tag{3.2}$$

Then we have

$$\inf_{\lambda\in(0,\infty)} g_{r,u}(\lambda) = g(\lambda_0) = \frac{r+u}{u^{\frac{u}{r+u}}\cdot r^{\frac{u}{r+u}}} C^{\frac{r}{r+u}} D^{\frac{r}{r+u}}. \tag{3.3}$$

Proof. We observe that

$$g'_{r,u}(\lambda) = \frac{rD\lambda^{r+u} - Cu}{\lambda^{u+1}}, \quad \lambda \in (0,\infty).$$

The unique solution of the equation $g'_{r,u}(\lambda) = 0$, $\lambda \in (0,\infty)$ is λ_0 provided by (3.2).

The function $g_{r,u}$ is decreasing on $(0,\lambda_0)$ and increasing on (λ_0,∞). The global minimum for $g_{r,u}$ on $(0,\infty)$ is

$$\begin{aligned} g_{r,u}(\lambda_0) &= \frac{C}{\left(\frac{uC}{rD}\right)^{\frac{u}{r+u}}} + D\left(\frac{uC}{rD}\right)^{\frac{r}{r+u}} \\ &= \frac{r+u}{u^{\frac{u}{r+u}} r^{\frac{r}{r+u}}} C^{\frac{r}{r+u}} D^{\frac{u}{r+u}} \end{aligned}$$

and the equality (3.3) is proved. □

The following particular cases are useful in applications.

Corollary 1. *Let $C,D>0$.*

(i) *For $r\in(0,1]$, consider the function $g_r:(0,\infty)\to(0,\infty)$, given by*

$$g_r(\lambda) = \frac{C}{\lambda} + D\lambda^r. \tag{3.4}$$

Define

$$\overline{\lambda_0} = \left(\frac{C}{rD}\right)^{\frac{1}{r+1}} \in (0,\infty). \tag{3.5}$$

Then we have

$$\inf_{\lambda\in(0,\infty)} g_r(\lambda) = g_r\left(\overline{\lambda_0}\right) = \frac{r+1}{r^{\frac{r}{r+u}}} C^{\frac{r}{r+1}} D^{\frac{1}{r+1}}. \tag{3.6}$$

(ii) *For $u\in(0,1]$, consider the function $g_u:(0,\infty)\to(0,\infty)$, given by*

$$g_u(\lambda) = \frac{C}{\lambda^u} + D\lambda. \tag{3.7}$$

Define

$$\widetilde{\lambda_0} = \left(\frac{uC}{D}\right)^{\frac{1}{1+u}} \in (0,\infty).$$

Then we have

$$\inf_{\lambda\in(0,\infty)} g_u(\lambda) = g_u\left(\widetilde{\lambda_0}\right) = \frac{1+u}{u^{\frac{u}{1+u}}} C^{\frac{1}{u+1}} D^{\frac{u}{u+1}}.$$

The following result holds.

Theorem 3. *Let J be an unbounded subinterval of $\mathbb{R}$ and $g:J\to\mathbb{R}$ a locally absolutely continuous function on J. If $g\in L_\infty(J)$, the derivative $g':J\to\mathbb{R}$ is of locally bounded variation and there exists a constant $V_J>0$ and $r\in(0,1]$ such that*

$$\left|\bigvee_a^b (g')\right| \le V_J |a-b|^r \quad \text{for any } \ a,b\in J; \tag{3.8}$$

then $g'\in L_\infty(J)$ and one has the inequality

$$\|g'\|_{J,\infty} \le \frac{2^{\frac{r}{r+1}}(r+1)}{r^{\frac{r}{r+1}}} \|g\|_{J,\infty}^{\frac{r}{r+1}} V_J^{\frac{1}{r+1}}. \tag{3.9}$$

Proof. Applying Theorem 1 for the function $f = g'$ on $[a,b]$ (or $[b,a]$), we deduce

$$\left\|g'\right\|_{[a,b],\infty} \leq \frac{|g(b) - g(a)|}{|b-a|} + \left|\bigvee_a^b (g')\right|. \tag{3.10}$$

for any $a, b \in J$, $a \neq b$.

Since $|g'(b)| \leq \|g'\|_{[a,b],\infty}$, $|g(b) - g(a)| \leq 2\|g\|_{J,\infty}$, then by (3.8) and (3.10) we deduce

$$\left|g'(b)\right| \leq \frac{2\|g\|_{J,\infty}}{|b-a|} + V_J |b-a|^r \tag{3.11}$$

for any $a, b \in J$, $a \neq b$.

Fix $b \in J$. Then for any $\lambda > 0$, there exists an $a \in J$ such that $\lambda = |b-a|$. Consequently, by (3.11), we deduce that

$$\left|g'(b)\right| \leq \frac{2\|g\|_{J,\infty}}{\lambda} + V_J \lambda^r \tag{3.12}$$

for any $\lambda > 0$ and $b \in J$.

Taking the infimum over $\lambda \in (0,\infty)$ in (3.12) and using Corollary 1, we deduce

$$\begin{aligned} \left|g'(b)\right| &\leq \frac{r+1}{r^{\frac{r}{r+1}}} \left(2\|g\|_{J,\infty}\right)^{\frac{r}{r+1}} \cdot V_J^{\frac{1}{r+1}} \\ &= \frac{2^{\frac{r}{r+1}}(r+1)}{r^{\frac{r}{r+1}}} \|g\|_{J,\infty}^{\frac{r}{r+1}} V_J^{\frac{1}{r+1}} \end{aligned} \tag{3.13}$$

for any $b \in J$. Finally, taking the supremum in (3.13) over $b \in J$, we deduce the desired result (3.9). □

There are a number of particular cases of interest.

Corollary 2. *Assume that $g : J \to \mathbb{R}$ is such that $g' : J \to \mathbb{R}$ is locally absolutely continuous and $g'' \in L_\infty(J)$. If $g \in L_\infty(J)$, then $g' \in L_\infty(J)$ and*

$$\left\|g'\right\|_{J,\infty} \leq 2\sqrt{2}\,\|g\|_{J,\infty}^{\frac{1}{2}} \left\|g''\right\|_{J,\infty}^{\frac{1}{2}}. \tag{3.14}$$

Proof. If $g'' \in L_\infty(J)$, then

$$\left|\bigvee_a^b (g')\right| = \left|\int_a^b \left|g''(t)\right| dt\right| \leq |b-a| \left\|g''\right\|_{J,\infty}$$

for any $a, b \in J$, giving, by (3.11), that

$$\left|g'(b)\right| \leq \frac{2\|g\|_{J,\infty}}{|b-a|} + \left\|g''\right\|_{J,\infty} |b-a| \tag{3.15}$$

for any $a, b \in J$, $a \neq b$.

Applying Theorem 3 for $V_J = \|g''\|_{J,\infty}$ and $r = 1$, we deduce (3.14). □

The following result is also of interest.

Corollary 3. *Assume that* $g : J \to \mathbb{R}$ *is such that* $g' \in L_p(J)$, $p > 1$. *If* $g \in L_\infty(J)$, *then* $g' \in L_\infty(J)$ *and*

$$\left\| g' \right\|_{J,\infty} \leq \frac{2^{\frac{p-1}{2p-1}} (2p-1)}{(p-1)^{\frac{p-1}{2p-1}} p^{\frac{p}{2p-1}}} \cdot \| g \|_{J,\infty}^{\frac{p-1}{2p-1}} \left\| g'' \right\|_{J,p}^{\frac{p-1}{2p-1}}. \tag{3.16}$$

Proof. Using Hölder's inequality, we have

$$\begin{aligned} \left| \bigvee_a^b (g') \right| &= \left| \int_a^b \left| g''(t) \right| dt \right| \leq \left| \int_a^b dt \right|^{\frac{1}{q}} \left| \int_a^b \left| g''(t) \right|^p dt \right|^{\frac{1}{p}} \\ &\leq |b-a|^{\frac{1}{q}} \left\| g'' \right\|_{J,p}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

for any $a, b \in J$, giving, by (3.11), that

$$\left| g'(b) \right| \leq \frac{2 \| g \|_{J,\infty}}{|b-a|} + |b-a|^{\frac{1}{q}} \left\| g'' \right\|_{J,p}, \tag{3.17}$$

for any $a, b \in J$, $a \neq b$.

Applying Theorem 3 for $V_J = \| g'' \|_{J,p}$ and $r = \frac{1}{q} = \frac{p-1}{p}$, we deduce (3.16). □

The following result also holds.

Theorem 4. *Let* J *be an unbounded subinterval of* $\mathbb{R}$ *and* $g : J \to \mathbb{R}$ *a locally absolutely continuous function on* J. *If* $g' \in L_1(J)$, *the derivative* $g' : J \to \mathbb{R}$ *is of locally bounded variation and there exists a constant* $V_J > 0$ *and* $r \in (0, 1]$ *such that*

$$\left| \bigvee_a^b (g') \right| \leq V_J |a-b|^r \quad \text{for any} \ \ a, b \in J; \tag{3.18}$$

then $g' \in L_\infty(J)$ *and one has the inequality*

$$\left\| g' \right\|_{J,\infty} \leq \frac{r+1}{r^{\frac{r}{r+1}}} \left\| g' \right\|_{J,1}^{\frac{r}{r+1}} V_J^{\frac{1}{r+1}}. \tag{3.19}$$

Proof. Since, for any $a, b \in J$,

$$|g(b) - g(a)| \leq \left| \int_a^b g'(s)\, ds \right| \leq \left| \int_a^b \left| g'(s) \right| ds \right| \leq \left\| g' \right\|_{J,1},$$

then, by (3.10) and (3.18), we deduce

$$\left| g'(b) \right| \leq \frac{\| g' \|_{J,1}}{|b-a|} + V_J |b-a|^r$$

for any $a, b \in J$, $a \neq b$.

Using an argument similar to the one in Theorem 3, we deduce (3.19). □

The following particular case also holds.

Corollary 4. *Assume that $g : J \to \mathbb{R}$ is such that $g' : J \to \mathbb{R}$ is locally absolutely continuous and $g'' \in L_\infty(J)$. If $g' \in L_1(J)$, then $g' \in L_\infty(J)$ and*

$$\left\|g'\right\|_{J,\infty} \le 2\left\|g'\right\|_{J,1}^{\frac{1}{2}}\left\|g''\right\|_{J,\infty}^{\frac{1}{2}}. \tag{3.20}$$

Corollary 5. *Assume that $g : J \to \mathbb{R}$ is such that $g' : J \to \mathbb{R}$ is locally absolutely continuous and $g'' \in L_p(J)$, $p > 1$. If $g' \in L_1(J)$, then $g' \in L_\infty(J)$ and*

$$\left\|g'\right\|_{J,\infty} \le \frac{2p-1}{(p-1)^{\frac{p-1}{2p-1}}\, p^{\frac{p}{2p-1}}} \cdot \|g\|_{J,1}^{\frac{p-1}{2p-1}}\left\|g''\right\|_{J,p}^{\frac{p-1}{2p-1}}. \tag{3.21}$$

We may state the following result as well.

Theorem 5. *Let J be an unbounded subinterval of $\mathbb{R}$ and $g : J \to \mathbb{R}$ a locally absolutely continuous function on J. If $g' \in L_\alpha(J)$, $\alpha > 1$, the derivative $g' : J \to \mathbb{R}$ is of locally bounded variation on J and there exists a constant $V_J > 0$ and $r \in (0,1]$ such that*

$$\left|\bigvee_a^b (g')\right| \le V_J\,|b-a|^r \quad \text{for any} \quad a, b \in J; \tag{3.22}$$

then $g' \in L_\infty(J)$ and one has the inequality

$$\left\|g'\right\|_{J,\infty} \le \frac{\alpha r+1}{\alpha^{\frac{\alpha r}{\alpha r+1}}\, r^{\frac{\alpha r}{\alpha r+1}}}\left\|g'\right\|_{J,\alpha}^{\frac{\alpha r}{\alpha r+1}} V_J^{\frac{1}{\alpha r+1}}. \tag{3.23}$$

Proof. By Hölder's integral inequality, we have

$$\begin{aligned}|g(b)-g(a)| &= \left|\int_a^b g'(s)\,ds\right| \le \left|\int_a^b \left|g'(s)\right| ds\right| \\ &\le |b-a|^{\frac{1}{\beta}}\left\|g'\right\|_{J,\alpha}, \quad \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\end{aligned}$$

and then, by (3.10) and (3.18), we deduce

$$\begin{aligned}\left|g'(b)\right| &\le \frac{|b-a|^{\frac{1}{\beta}}\left\|g'\right\|_{J,\alpha}}{|b-a|} + |b-a|^r V_J \\ &= \frac{\left\|g'\right\|_{J,\alpha}}{|b-a|^{\frac{1}{\alpha}}} + |b-a|^r V_J\end{aligned} \tag{3.24}$$

for any $a, b \in J$, $a \ne b$.

Fix $b \in J$. Then for any $\lambda > 0$, there exists an $a \in J$ such that $\lambda = |b-a|$. Consequently, by (3.14) we deduce that

$$\left|g'(b)\right| \le \frac{\left\|g'\right\|_{J,\alpha}}{\lambda^{\frac{1}{\alpha}}} + \lambda^r V_J \tag{3.25}$$

for any $\lambda > 0$ and $b \in J$.

Taking the infimum over $\lambda \in (0,\infty)$ in (3.25) and using Lemma 1 for $u = \frac{1}{\alpha}$, we deduce

$$\begin{aligned} \left|g'(b)\right| &\leq \frac{r+\frac{1}{\alpha}}{\left(\frac{1}{\alpha}\right)^{\frac{\frac{1}{\alpha}}{r+\frac{1}{\alpha}}} r^{\frac{r}{r+\frac{1}{\alpha}}}} \left\|g'\right\|_{J,\alpha}^{\frac{r}{r+\frac{1}{\alpha}}} V_J^{\frac{\frac{1}{\alpha}}{r+\frac{1}{\alpha}}} \\ &= \frac{\alpha r+1}{\alpha^{\frac{\alpha r}{\alpha r+1}} r^{\frac{\alpha r}{\alpha r+1}}} \left\|g'\right\|_{J,\alpha}^{\frac{\alpha r}{\alpha r+1}} V_J^{\frac{1}{\alpha r+1}} \end{aligned}$$

for any $b \in J$, giving the desired result (3.23). □

The following corollary holds.

Corollary 6. *Assume that $g : J \to \mathbb{R}$ is such that g' is locally absolutely continuous and $g'' \in L_\infty(J)$. If $g' \in L_\alpha(J)$, $\alpha > 1$, then $g' \in L_\infty(J)$ and*

$$\left\|g'\right\|_{J,\infty} \leq \frac{\alpha+1}{\alpha^{\frac{\alpha}{\alpha+1}} r} \left\|g'\right\|_{J,\alpha}^{\frac{\alpha}{\alpha+1}} \left\|g''\right\|_{J,\infty}^{\frac{1}{\alpha+1}}. \tag{3.26}$$

Finally we have

Corollary 7. *Assume that $g : J \to \mathbb{R}$ is such that g' is locally absolutely continuous and $g'' \in L_p(J)$, $p > 1$. If $g' \in L_\alpha(J)$, $\alpha > 1$, then $g' \in L_\infty(J)$ and*

$$\left\|g'\right\|_{J,\infty} \leq \frac{\alpha(p-1)+p}{\alpha^{\frac{\alpha(p-1)}{\alpha(p-1)+p}} (p-1)^{\frac{\alpha(p-1)}{\alpha(p-1)+p}} \cdot p^{\frac{p}{\alpha(p-1)+p}}} \left\|g'\right\|_{J,\alpha}^{\frac{\alpha(p-1)}{\alpha(p-1)+p}} \left\|g''\right\|_{J,p}^{\frac{p}{\alpha(p-1)+p}}. \tag{3.27}$$

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